① a) $P(x) = -x^3 + x^2 - x + 1$

$P(-1) = 1 + 1 + 1 + 1$ No

b) $2 \cdot \frac{1}{4} - 3 \cdot \frac{1}{2} + 1 = \frac{1}{2} - \frac{3}{2} + 1 = 0$ yes.

② a) $x = 0 \quad m = 2$

$x = -1 \quad m = 3$

$x = 1 \quad m = 1$

b) $x = 0 \quad m = 2$

$x = -1 + \sqrt{2} \quad m = 1$

$x = -\frac{5}{2} \quad m = 1$

c) $x = \frac{2}{7} \quad m = 3$

$x^2 = -9$

$x = \pm 3i \quad m = 1$

③

$$
\begin{array}{r|l}
 & 5x^2 - 16x + 4 \\
\hline
x+2 & 5x^3 - 6x^2 - 28x - 2 \\
 & -5x^3 - 10x^2 \\
\cline{2-2}
 & \quad -16x^2 - 28x - 2 \\
 & \quad +16x^2 + 32x \\
\cline{2-2}
 & \qquad 4x - 2 \\
 & \qquad -4x - 8 \\
\cline{2-2}
 & \qquad \textcircled{-10}
\end{array}
$$

④

$$
\begin{array}{r|l}
 & x^2 + 5x + 15 \\
\hline
x-3 & x^3 + 2x^2 + \textcircled{0}x - 4 \\
 & -x^3 + 3x^2 \\
\cline{2-2}
 & \quad 5x^2 + 0x - 4 \\
 & \quad -5x^2 + 15x \\
\cline{2-2}
 & \qquad 15x - 4 \\
 & \qquad -15x + 45 \\
\cline{2-2}
 & \qquad \textcircled{41}
\end{array}
$$

⑤

$$
\begin{array}{c|cccc}
 & 2 & -3 & 4 & -5 \\
\hline
2 & \underbrace{2 \quad 1 \quad 6}_{\text{quotient}} & & & \textcircled{7}\ R
\end{array}
$$

$f(2) = 2(8) - 3(4) + 4(2) - 5 = 16 - 12 + 8 - 5 = 7$

$f(2) = 7 =$ remainder

⑥ $f(x) = x^4 - 6x^2 + 4x - 8$

$f(-3) = ?$

$$
\begin{array}{c|ccccc}
 & 1 & 0 & -6 & 4 & -8 \\
\hline
-3 & 1 & -3 & 3 & -5 & \textcircled{7}
\end{array}
$$

$f(-3) =$ remainder when dividing $f(x)$ by $x+3$

⑦ $f(x) = 2x^3 - 4x^2 + 2x - 1$

$f(x) \div (x-2)$ < long division

synthetic division

$f(2) = r = 2(8) - 4(4) + 2(2) - 1 = 3$

⑧ $f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11$

$x+2 \mid f(x)$ iff $f(-2) = 0$

$f(-2) = -8k + 4 - 2k^2 + 3k^2 + 11 = 0$

$k^2 - 8k + 15 = 0$ $k = 3$

$(k-3)(k-5) = 0$ < or

$k = 5$

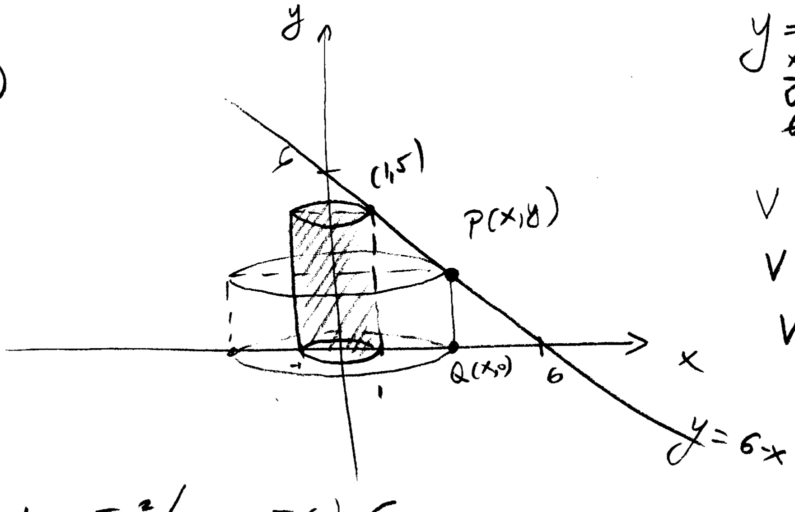
⑨ $f(x) = 3x^4 + x^2 + 5$

$x - c \mid f(x)$ iff $f(c) = 0$

$f(c) = 3c^4 + c^2 + 5 > 0$ $\quad x-c \nmid f(x)$

$f(c) \neq 0 \Rightarrow$

⑩



$y = 6 - x$

| $x$ | $y$ |
|---|---|
| 0 | 6 |
| 6 | 0 |

$V = A_{base} \cdot \text{height}$

$V = \pi r^2 \cdot h$

$V = \pi x^2 (y)$

$$\boxed{V = \pi x^2 (6-x)}$$

$x \in (0,6)$

$V_2 = \pi r^2 h = \pi(1) \cdot 5$

$V_2 = 5\pi$

want $V = V_2$

$\pi x^2 (6-x) = 5\pi$

$x^2(6-x) = 5$

$6x^2 - x^3 = 5$

$x^3 - 6x^2 + 5 = 0$ $\quad x \neq 1$

Note that $x=1$ is a zero

$$\begin{array}{c|cccc} & 1 & -6 & 0 & 5 \\ 1 & 1 & -5 & -5 & 0 \end{array}$$

$(x-1)(x^2 - 5x - 5) = 0$ < $x = 1$

$x = \frac{5 \pm \sqrt{45}}{2}$

want $x > 0$

so $x = \frac{5+\sqrt{45}}{2}$ $\quad y = 6 - x = 6 - \frac{5+\sqrt{45}}{2}$

$$P(x,y) = \left( \frac{5+\sqrt{45}}{2}, \frac{7-\sqrt{45}}{2} \right)$$

---

⑪ $f(x) = 3x^2 - 5x - 2$

a) $x = \frac{5 \pm \sqrt{25+24}}{6} = \frac{5 \pm 7}{6} \left\langle \begin{matrix} 2 \\ \frac{-1}{3} \end{matrix} \right.$ $\quad \boxed{f(x) = 3(x-2)(x+\frac{1}{3})}$

Recall $f(x) = (x-2)(3x+1)$

b) $x^2 - 5 = 0$

$x = \pm \sqrt{5}$ $\quad f(x) = (x+\sqrt{5})(x-\sqrt{5})$

c) $x^2 - 4x + 5 = 0$ $\quad f(x) = (x-2-i)(x-2+i)$

$x = 2+i$

$x = 2-i$

---

⑫ ⓐ $f(x) = 4x^5 + 12x^4 + 9x^3$

zeros: $4x^5 + 12x^4 + 9x^3 = 0$

$x^3(4x^2 + 12x + 9) = 0$

$4x^2 + 12x + 9 = 0$

$(2x+3)^2 = 0$

$$\begin{cases} x = 0 & m = 3 \\ x = \frac{-3}{2} & m = 2 \end{cases}$$

$f(x) = 4(x^3)(x + \frac{3}{2})^2$

ⓑ $f(x) = x^4 + 21x^2 - 100$

zeros: $x^4 + 21x^2 - 100 = 0$

let $x^2 = t$

$t^2 + 21t - 100 = 0$ $\left\langle \begin{matrix} t = -25 \\ t = 4 \end{matrix} \right.$

$(t+25)(t-4) = 0$

$x^2 = -25$ $\quad x^2 = 4$

$x = \pm 5i$ m(1) $\quad x = \pm 2$ (m=1)

$f(x) = (x-5i)(x+5i)(x-2)(x+2)$

⑬ ⓐ $x=-3$ zero iff $f(-3)=0$ (remainder when dividing by $x+3$)

$$\begin{array}{r|rrrr} & 6 & 13 & -14 & 3 \\ \hline -3 & 6 & -5 & 1 & \boxed{0} \end{array}$$

$f(x)=(x+3)(6x^2-5x+1)$

$6x^2-5x+1=0$

$$x=\frac{5\pm\sqrt{25-24}}{12}=\frac{5\pm1}{12}\left\langle\begin{matrix}\frac{1}{2}\\ \frac{1}{3}\end{matrix}\right.$$

$f(x)=6(x+3)(x-\frac{1}{2})(x-\frac{1}{3})$

OR

$f(x)=(x+3)(2x-1)(3x-1)$

ⓑ $f(x)=x^4-9x^3+22x^2-32$

$x=4$ zero of $m=4 \iff (x-4)^2 \mid f(x)$

Method I $\quad (x-4)^2=x^2-8x+16 \mid f(x)$

Long division

$$\begin{array}{r l} & \quad x^2 - x - 2 \\ \cline{2-2} x^2-8x+16 & \big)\, x^4-9x^3+22x^2+0x-32 \\ & \;\, -x^4+8x^3-16x^2 \\ \cline{2-2} & \qquad\; -x^3+6x^2+0x-32 \\ & \qquad\;\;\; x^3-8x^2+16x \\ \cline{2-2} & \qquad\qquad -2x^2+16x-32 \\ & \qquad\qquad\;\; 2x^2-16x+32 \\ \cline{2-2} & \qquad\qquad\qquad\qquad 0 \end{array}$$

$f(x)=(x-4)^2(x^2-x-2)$

$x^2-x-2=0$

$x=2$ OR $x=-1$

$f(x)=(x-4)^2(x-2)(x+1)$

Method II

Synthetic division

$$\begin{array}{r|rrrrr} & 1 & -9 & 22 & 0 & -32 \\ \hline 4 & 1 & -5 & 2 & 8 & 0 \end{array}$$

$f(x)=(x-4)(x^3-5x^2+2x+8)$

$$\begin{array}{r|rrrr} & 1 & -5 & 2 & 8 \\ \hline 4 & 1 & -1 & -2 & 0 \end{array}$$

$f(x)=(x-4)^2(x^2-x-2)=(x-4)^2(x-2)(x+1)$

(14) degree $f(x) = 3$

$x=-1$
$x=2$
$x=3$
zeros iff
$x+1 \mid f(x)$
$x-2 \mid f(x)$
$x-3 \mid f(x)$

So, $f(x) = a(x+1)(x-2)(x-3)$

$f(-2)=80 \Rightarrow a(-1)(-4)(-5) = 80$

$-20a = 80$

$a = -4$

$f(x) = -4(x+1)(x-2)(x-3)$

(15) degree $f(x) = 4$, $a=1$

$x=-5$ zero $m=2$ iff $(x+5)^2 \mid f(x)$

$x=2$ zero $m=2$ iff $(x-2)^2 \mid f(x)$

So, $f(x) = (x+5)^2(x-2)^2$

| $x$ | $-\infty$ | | $-5$ | | $0$ | | $2$ | | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $\infty$ | $+$ | $0$<br>$m=2$ | $+$ | $100$ | $+$ | $0$<br>$m=2$ | $+$ | $\infty$ |

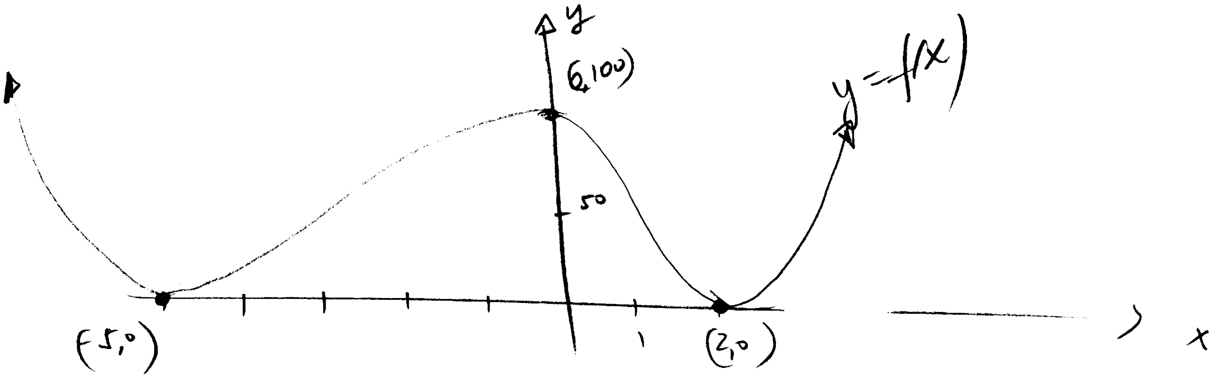
Domain: $x \in R$

x-int: $(-5,0)$ and $(2,0)$

y-int: when $x=0$, $y = 5^2(-2)^2 = 100$ $(0,100)$

End behavior: given by $x^2x^2 = x^4$

when $x \to \pm\infty$, $y \to \infty$



⑯ degree $f(x) = 3$

x-n: $x = 1$ zero of $m =$ even
$x = 3$ zero of $m =$ odd $\Big\} \Rightarrow f(x) = a(x-1)^2(x-3)$

y-n: $x = 0, y = 3$

$a(-1)^2(-3) = 3$

$-3a = 3$

$a = -1$

$f(x) = -(x-1)^2(x-3)$

⑰ (a) $f(x) = \underbrace{4x^3 - }_{1}\underbrace{6x^2}_{2}\underbrace{+x}_{3} - 3$

There are 3 variations in sign in $f(x)$
$\Rightarrow$ there are 3 or 1 positive real zeros

$f(-x) = -4x^3 - 6x^2 - x - 3$

There are no variations in sign in $f(-x)$
$\Rightarrow$ there are 0 negative real zeros

$f(x)$ can have either 3 positive real zeros

OR $\Big\{$ 1 positive real zero
2 imaginary zeros (nonreal complex)

(b) $f(x) = 3x^4 + \underbrace{2x^3 - }_{1}\underbrace{4x}_{2} + 2$

There are 2 variations in sign in $f(x) \Rightarrow$ 2 or 0 positive zeros

$f(-x) = \underbrace{3x^4 - 2x^3}_{}\underbrace{+ 4x}_{2} + 2$

There are 2 variations in sign in $f(-x) \Rightarrow$ 2 or 0 negative zeros

**Possibilities**

① $\Big\{$ 2 + zeros
2 − zeros

(2) $\Big\{$ 2 + zeros
2 nonreal complex

(3) $\Big\{$ 2 − zeros
2 nonreal complex

(4) $\big\{$ 4 nonreal complex

⑱ $2x^3 - 5x^2 + 4x - 8 = 0$

Let $f(x) = 2x^3 - 5x^2 + 4x - 8$

Divide $f(x)$ by $x-1$

$$\begin{array}{r|rrrr} & 2 & -5 & 4 & -8 \\ \hline 1 & 2 & -3 & 1 & -7 \end{array}$$

can't apply 1st Theorem on Bounds

Divide $f(x)$ by $x-2$

$$\begin{array}{r|rrrr} & 2 & -5 & 4 & -8 \\ \hline 2 & 2 & -1 & & \end{array}$$

can't apply 1st Theorem on Bounds.

Divide $f(x)$ by $x-3$

$$\begin{array}{r|rrrr} & 2 & -5 & 4 & -8 \\ \hline 3 & 2 & 1 & 7 & 13 \end{array}$$

All #'s are +, therefore $x=3$ is the upper bound of the real zeros of $f(x)$

( $c < 3$, any $c$ = zero of $f(x)$)

Divide $f(x)$ by $x+1$

$$\begin{array}{r|rrrr} & 2 & -5 & 4 & -3 \\ \hline -1 & 2 & -7 & 11 & -14 \end{array}$$

#'s alternate, therefore $x = -1$ is the lower bound of the real zeros of $f(x)$

( $c > -1$, any $c$ = zero of $f(x)$)

So, If $c$ = zero of $f(x)$,

$-1 < c < 3$

(19) x-n : $(-3,0)$, so $x=-3$ zero with $m=\text{odd} \geq 3$

$(-1,0)$ so $x=-1$ zero with $m=\text{odd}$

$(2,0)$ so $x=2$ zero with $m=\text{even}$

want degree $f(x)$ = minimum

so $f(x) = a(x+3)^3(x+1)(x-2)^2$

b) $a=2$, $f(x)= 2(x+3)^3(x+1)(x-2)^2$

y-n: at $x=0$, $y = 2(3^3)(1)(-2)^2$

$y = 2(27)(4) = 216$

$a=1$, y-n $(0,108)$

---

(20) $t$ = time (in hours) - independent variable

$T$ = temp (°C) - dependent variable

degree $f(t)=4$

t-n: $(0,0)$ $t=0$ is a zero of $f(t)$

$(5,0)$ $t=5$ — " ——

$(19,0)$ $t=19$ — " ——

$(24,0)$ $t=24$ — " ——

$f(t)= at(t-5)(t-19)(t-24)$

want $(12,10) \in$ graph ; when $t=12$, $T=10$

$10 = f(12)$

$10 = 12a(7)(-7)(-12)$

$10 = 7056a$

$a = \frac{10}{7056} = \frac{5}{3528}$

$f(t)= \frac{5}{3528} t(t-5)(t-19)(t-24)$

㉑ $f(x) = x^3 - 7x^2 + 17x - 15$

$x = 2-i$ zero

All coefficients of $f(x) \in \mathbb{R}$, therefore

$x = 2+i$ is also a zero

$$\left.\begin{array}{lll} x=2-i \text{ zero} & \text{iff} & x-(2-i) \mid f(x) \\ x=2+i \text{ zero} & \text{iff} & x-(2+i) \mid f(x) \end{array}\right\} \Rightarrow$$

$$(x-2+i)(x-2-i) \mid f(x)$$

$$(x-2)^2 - i^2 \mid f(x)$$

$$x^2-4x+4+1 \mid f(x)$$

$$x^2-4x+5 \mid f(x)$$

$$\begin{array}{r|l} & x-3 \\ \hline x^2-4x+5 & x^3-7x^2+17x-15 \\ & -x^3+4x^2-5x \\ \hline & \quad -3x^2+12x-15 \\ & \quad +3x^2-12x+15 \\ \hline & \qquad 0 \end{array}$$

$f(x) = (x^2-4x+5)(x-3)$

zeros are $\begin{cases} x = 2-i & m=1 \\ x = 2+i & m=1 \\ x = 3 & m=1 \end{cases}$

(22) $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$

a) possible rational zeros

$$\frac{p}{q} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6}$$

$$\frac{p}{q} \in \left\{\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}\right\}$$

(b) Note that $f(1) = 0 \Rightarrow$ $x = 1$ zero

$x - 1 \mid f(x)$

$$\begin{array}{r|rrrrr} & 6 & 7 & -12 & -3 & 2 \\ \hline 1 & 6 & 13 & 1 & -2 & 0 \end{array}$$

$$f(x) = (x-1)(6x^3 + 13x^2 + x - 2)$$

we want to factor $6x^3 + 13x^2 + x - 2$

Try $x = -1$ $\quad -6 + 13 - 1 - 2 \neq 0$

Try $x = -2$

$$\begin{array}{r|rrrr} & 6 & 13 & 1 & -2 \\ \hline -2 & 6 & 1 & -1 & 0 \end{array}$$

so $x + 2 \mid 6x^3 + 13x^2 + x - 2$

$x = -2$

$$f(x) = (x-1)(x+2)(6x^2 + x - 1)$$

$6x^2 + x - 1 = 0$

$x = \frac{1}{3}$ or $x = \frac{-1}{2}$

The rational zeros are
$$\left\{ \begin{array}{ll} x = 1 & m = 1 \\ x = -2 & m = 1 \\ x = \frac{1}{3} & m = 1 \\ x = \frac{-1}{2} & m = 1 \end{array} \right.$$

$$f(x) = 6(x-1)(x+2)\left(x - \tfrac{1}{3}\right)\left(x + \tfrac{1}{2}\right)$$

or

$$f(x) = (x-1)(x+2)(3x-1)(2x+1)$$

㉓ (a) $f(x) = x^3 + \underset{1}{6x^2} - x - 30$

(i) max # real zeros = 3

(ii) There is 1 variation in sign in $f(x)$ → 1 positive real zero

$f(-x) = \underbrace{-x^3 + 6x^2}_{1} \underbrace{+ x - 30}_{2}$

There are 2 variations in sign in $f(-x)$ ⇒ 2 or 0 negative real zeros

(iii) possible rational zeros

$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30}{\pm 1}$$

$$\frac{p}{q} \in \pm\{1, 2, 3, 5, 6, 10, 15, 30\}$$

(iv) Note that $f(1) \neq 0$, so $x = 1$ not a zero

try $x = 2$

| | | | | |
|---|---|---|---|---|
| | 1 | 6 | $-1$ | $-30$ |
| 2 | 1 | 8 | 15 | 0 |

so $x = 2$ zero and

$f(x) = (x-2)(x^2 + 8x + 15)$

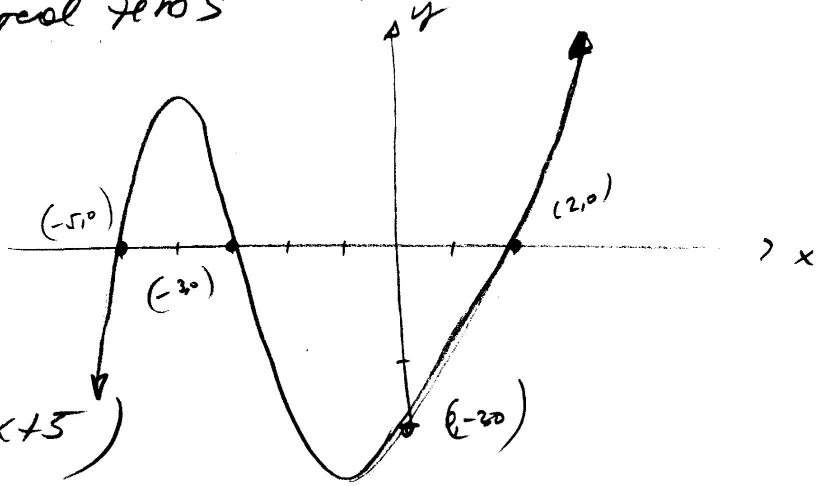
Also, note that $x = 2$ upper bound for the real zeros

$x^2 + 8x + 15 = 0$

$x = -3$ or $x = -5$

The zeros of $f(x)$:

$\begin{cases} x = 2 & m = 1 \\ x = -5 & m = 1 \\ x = -3 & m = 1 \end{cases}$



(v) $f(x) = (x-2)(x+3)(x+5)$

(vi) Domain: $x \in \mathbb{R}$

| x | $-\infty$ | | $-5$ | | $-3$ | | 0 | | 2 | | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| f(x) | $-\infty$ | − | 0 | + | 0 | − | $-30$ | − | 0 | + | $\infty$ |
| | | | m=1 | | m=1 | | | | m=1 | | |

x-in: (2,0), (-3,0), (-5,0)

y-in: (0,-30)

End-behavior = given by $x^3$: when $x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$

(23) (b) $f(x) = 6x^5+19x^4+x^3-6x^2$

(i) max # real zeros = 5

To apply Descartes' Rule of Signs and Rational Zeros Theorem, we need constant term ≠ 0

$$\underline{f(x) = x^2(6x^3+19x^2+x-6)}$$

Let $\boxed{g(x) = 6x^3+19x^2+\underbrace{x-6}_{1}}$

(ii) g has one positive real zero

$$g(-x) = \underbrace{-6x^3+19x^2}_{1}\underbrace{\phantom{x}-x}_{2}-6$$

g has 2 or 0 negative real zeros

(iii) $\dfrac{p}{q} = \dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 3, \pm 6}$ for $g(x)$

$$\frac{p}{q} \in \pm\left\{1, 2, 3, 6, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, \tfrac{2}{3}, \tfrac{3}{2}\right\}$$

(iv) Rational zeros of $g(x) = 6x^3+19x^2+x-6$

Try $x=1$
$$\begin{array}{c|cccc} & 6 & 19 & 1 & -6 \\ \hline 1 & 6 & 25 & 26 & 20 \end{array}$$

note that $x=1$ not a zero, but $x=1$ upper bound

Try $x=\frac{1}{2}$
$$\begin{array}{c|cccc} & 6 & 19 & 1 & -6 \\ \hline \frac{1}{2} & 6 & 22 & 12 & 0 \end{array}$$

$x=\frac{1}{2}$ zero

$$g(x) = (x-\tfrac{1}{2})(6x^2+22x+12)$$
$$g(x) = 2(x-\tfrac{1}{2})(3x^2+11x+6)$$

$3x^2+11x+6=0$

$$x = \frac{-11 \pm \sqrt{121-72}}{6} = \frac{-11\pm 7}{6} \left\langle \begin{array}{l} x=-3 \\ \text{or} \\ x=-\frac{2}{3} \end{array}\right.$$

$g(x) = 2(x-\frac{1}{2})\,3\,(x+3)(x+\frac{2}{3})$

(iv) All rational zeros of $f(x)$ are

$$\begin{cases} x=0 & m=2 \\ x=\frac{1}{2} & m=1 \\ x=-3 & m=1 \\ x=-\frac{2}{3} & m=1 \end{cases}$$

(v) $f(x)=6x^2(x-\frac{1}{2})(x+3)(x+\frac{2}{3})$

OR

$f(x)= x^2(2x-1)(x+3)(3x+2)$

(vi) Domain $= \mathbb{R}$

| $x$ | $-\infty$ | | $-3$ | | $-\frac{2}{3}$ | | $0$ | | $\frac{1}{2}$ | | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-\infty$ | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $-$ | $0$ | $+$ | $\infty$ |
| | | | $m=1$ | | $m=1$ | | $m=2$ | | $m=1$ | | |
| | | | / | | / | | ⌣ | | / | | |

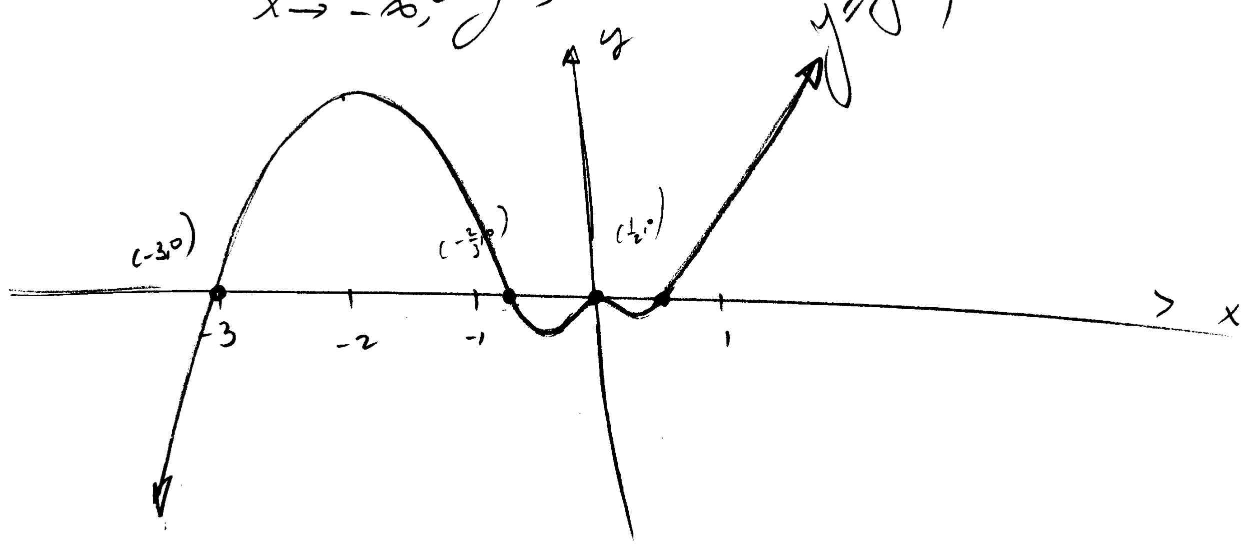
x-n: $(0,0)$, $(\frac{1}{2},0)$, $(-3,0)$, $(-\frac{2}{3},0)$

y-n: $(0,0)$

End behavior given by $6x^5$

when $x \to \infty$, $y \to \infty$

$x \to -\infty$, $y \to -\infty$



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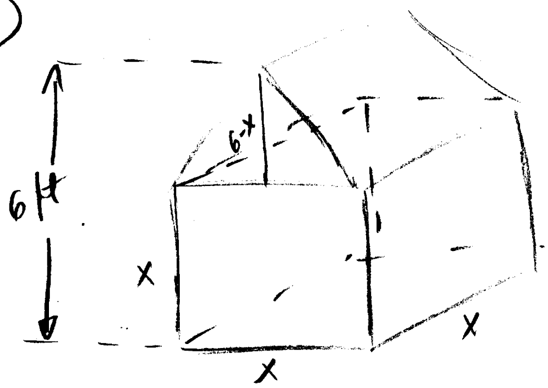

a) $V = V_{cube} + V_{prism}$

$V_{cube} = x^3$

$V_{prism} = A_{base} \cdot height$

$= A_{\Delta} \cdot x$

$= \frac{1}{2} x \cdot (6-x) \cdot x$

$V = x^3 + \frac{1}{2}x^2(6-x)$

b) $V = 80 \text{ ft}^3$

$x^3 + \frac{1}{2}x^2(6-x) = 80 \quad / \cdot 2$

$2x^3 + 6x^2 - x^3 = 160$

$x^3 + 6x^2 - 160 = 0$

let $f(x) = x^3 + 6x^2 - 160$

note $f(1) \neq 0$

$f(2) = 8 + 24 - 160 \neq 0$

$f(3) = 27 + 54 - 160 \neq 0$

try $x = 4$

$$\begin{array}{c|cccc} & 1 & 6 & 0 & -160 \\ \hline 4 & 1 & 10 & 40 & 0 \end{array}$$

$f(x) = (x-4)(x^2 + 10x + 40) = 0$

only $x = 4$ real solution