Sections 2.1, 2.2, and 2.3
What is a Function? Graphs of Functions
Properties of Functions

Definition

1. **A function** is a rule (relation) that assigns to each element x in a set A exactly one element , called $f(x)$, in a set B.

2. **A function** is a relation between two variables: **independent variable (input, argument)** and **dependent variable (output)** that assigns to each independent variable a unique value of the dependent variable.

Notes:
- We usually consider functions for which the sets A and B are sets of real numbers.
- The symbol $f(x)$ is read " f of x" or "f at x" and is called the **value of *f* at *x***, or the **image of *x* under *f*.**
- The set **A** is called the **domain** of the function.
- The **range** of a function f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Definition

If $y = f(x)$, the **domain** of f is the set of values for the independent variable, x

$$D_f = \{x \mid f(x) \in \mathbb{R}\}$$

and the **range** of f is the set of values for the dependent variable, y

$$R_f = \{y \mid y = f(x),\ x \in D_f\}$$

Definition

For a function f that is smooth and continuous on an interval containing a and b, the **average rate of change between *a* and *b* (from a to b)** is

$$\frac{f(b)-f(a)}{b-a}, \text{ where } a \neq b.$$

Definition

For a function $f(x)$ that is smooth and continuous on the interval containing x and $x+h$

where $h \neq 0$ (constant), the **difference quotient for *f*** is defined as

$$\frac{f(x+h)-f(x)}{h}$$

Functions defined by tables

1. The table shows the daily low temperature for a one-week period in New York during July.

Date in July	17	18	19	20	21	22	23
Low temperature ($^\circ F$)	73	77	69	73	75	75	70

a) What was the low temperature on July 19^{th}?
b) When was the low temperature $73^\circ F$?
c) Is the low temperature a function of the date?
d) Is the date a function of the low temperature?

2.

t	h
0	0
1	3
2	5
3	2
6	0

a) Is h a function of t? Explain.
b) Find $f(3),\ f(2),\ f(0),\ f(5)$
c) Solve $f(x)=0$ and $f(x)=6$.
d) Find the domain and range of the function.
e) Graph this function.

Functions defined by equations

3. Let $f(x)=\sqrt{x-1}$.

a) Express in words how f acts on the input x to produce the output $f(x)$.
b) Evaluate $f(1),\ f(3),\ f(a),\ f(x+2),\ f(-x), -f(x), f(x+h)$.
c) Find the domain and range of the function.
d) Find the intercepts of the graph (if any).
e) Is the point $(2,1)$ on the graph?
f) If $f(x)=3$, what is x? What point is on the graph?
g) Find the average rate of change from 2 to 5.
h) Find the difference quotient. Rationalize its numerator and simplify.

4. Let $g(x)=\dfrac{3}{x-2}$.

a) Find the domain and range.
b) Find $g(1), g(3), g(t), g(a+h), g(x^2), g(-x), -g(x), g(x+h)$.
c) Find the intercepts of the graph (if any).
d) Find the average rate of change from 1 to 5.
e) Find and simplify the difference quotient.

5. Let $h(x)=-3x^2+5x$.

a) Find the domain and intercepts.
b) Find $H(-1), H(-x), -H(x), H(x+h)$.
c) Find and simplify the difference quotient.

6. Let

$$f(x)=\begin{cases} x^2+2x, & \text{if } x\le -1 \\ x, & \text{if } -1<x\le 1 \\ -1, & \text{if } x>1 \end{cases}$$ a piecewise defined function.

a) What is the domain?
b) Find $f(-4),\ f\left(-\dfrac{3}{2}\right),\ f(-1),\ f(0),\ f(25)$

7. Let $f(x)=3x+2$ and $f(x)=\dfrac{2x}{x-1}$. For each function, find $\dfrac{f(a+h)-f(a)}{h}, h\ne 0$.

8. Let $s(t)=11t^2+t+100$ be the position , in miles, of a car driving on a straight road at time *t*, in hours.

The car's velocity at any time *t* is given by $v(t)=22t+1$.

a) Use function notation to express the car's position after 2 hours. Where is the car then?
b) Use function notation to express the question " When is the car going 65 mph?"
c) Where is the car when it is going 67 mph?

9. Determine whether the equation defines *y* as a function of *x*.

$x^2+y=9$, $x^2+(y-1)^2=4$

10. Find a function that represents the bottom half of the circle $x^2+y^2=9$.

The vertical line test:

A graph represents a function if and only if it passes the vertical line test, that is, any vertical line intersects the graph in at most one point.

Definition Suppose that a function *f* is defined over an interval *I*. If x_1 and x_2 are in *I*,

a) ***f*** **increases** on *I* iff, whenever $x_1<x_2,$ $f(x_1)<f(x_2)$;
b) ***f*** **decreases** on *I* iff, whenever $x_1<x_2,$ $f(x_1)>f(x_2)$;
c) ***f*** **is constant** on *I* iff, for every x_1 and x_2, $f(x_1)=f(x_2)$.

Definition A function f defined on some interval I has a **local maximum** (or **relative maximum**) at c iff $f(c)\geq f(x)$ for all *x* in some open interval containing *c*. The number $f(c)$ is called a **local maximum value** of f. Similarly, f has a **local minimum** (or **relative minimum**) at c iff $f(c)\leq f(x)$ for all *x* in some open interval containing *c*. The number $f(c)$ is called a **local minimum value** of f.

Definition Let f be a function with domain *D*. A function f has an **absolute maximum** (or **global maximum**) at c if f $f(c)\geq f(x)$ for all *x* in *D*. The number $f(c)$ is called the **maximum value** of *f* on *D*. Similarly, *f* has an **absolute minimum** at *c* if $f(c)\leq f(x)$ for all *x* in *D*. The number $f(c)$ is called the **minimum value** of *f* on *D*.

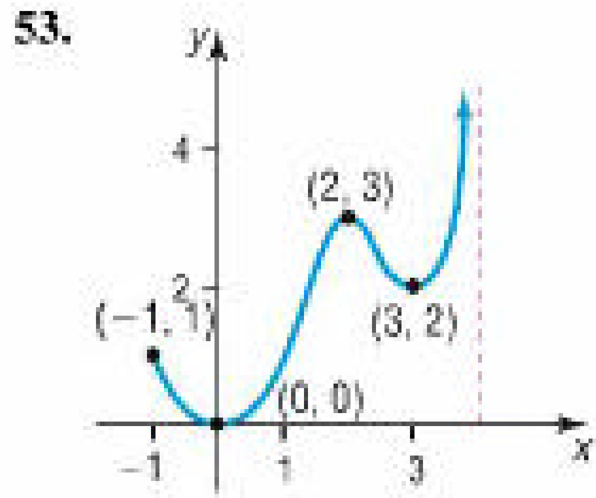
Definition A **graph is symmetric about the *y*-axis** if and only if , given any point (x,y) on the graph, the point $(-x,y)$ is also on the graph.

Definition A **graph is symmetric about the *x*-axis** if and only if , given any point (x,y) on the graph, the point $(x,-y)$ is also on the graph.

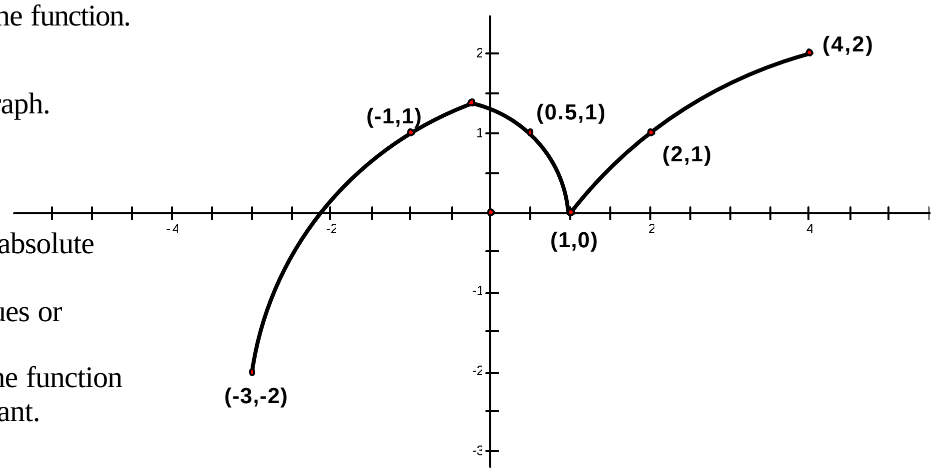
Definition A **graph is symmetric about the origin** if and only if , given any point (x,y) on the graph, the point $(-x,-y)$ is also on the graph.

Functions defined by graphs

11. a) Find the domain and range of the function.
b) Identify the intercepts on the graph.
c) Find $f(-3)$, $f(-1)$ and $f(4)$.
d) Find the absolute maximum and absolute minimum, if they exist.
e) Identify any local maximum values or local minimum values.
f) Identify the intervals on which the function is increasing, decreasing, and constant.



12.
a) Find the domain and range of the function.
b) Find $f(1)$.
c) Identify the intercepts on the graph.
d) Find all x such that $f(x)=1$.
e) Find all x such that $f(x)>1$.
f) Find the absolute maximum and absolute minimum, if they exist.
g) Identify any local maximum values or local minimum values.
h) Identify the intervals on which the function is increasing, decreasing, and constant.



13.
Find the domain of each function.

$$f(x)=x^2+1, g(x)=\frac{1}{3x-6}, h(x)=\frac{x^4}{x^2+x-6}, L(x)=\sqrt{7-3x}, l(x)=\frac{4}{\sqrt{x-9}}$$

Definition A **function is even** if and only if $f(-x)=f(x)$.

Definition A **function is odd** if and only if $f(-x)=-f(x)$

Theorem A **function is even** if and only if its graph is **symmetric with respect to the y-axis**.

A **function is odd** if and only if its graph is **symmetric with respect to the origin.**

14. Determine algebraically whether each function is even, odd, or neither.

$$f(x)=5x^4-x^2 \qquad g(x)=\frac{x}{x^2-1} \qquad h(x)=4x^3+5$$

The Algebra and Composition of Functions (2.1 & 5.1)

Two functions *f* and *g* can be combined to form new functions $f+g,\ f-g,\ fg, \dfrac{f}{g}$ in a manner similar to the way we add, subtract, multiply and divide real numbers.

<u>Definition</u> Let *f* and *g* be two functions. Let D_f be the domain of *f* and D_g the domain of *g*. Then:

- $(f+g)(x)=f(x)+g(x)$ and the domain of $f+g$ is $D_f \cap D_g$ (all real numbers that are common to the domain of *f* and the domain if *g*.)
- $(f-g)(x)=f(x)-g(x)$ and the domain of $f-g$ is $D_f \cap D_g$
- $(fg)(x)=f(x)\cdot g(x)$ and the domain of fg is $D_f \cap D_g$
- $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ and the domain of $\dfrac{f}{g}$ is the set of all real numbers that are common to the domain of *f* and the domain of *g* such that $g(x)\neq 0$

<u>Definition</u> If *f* and *g* are function, then the **composite function**, or **composition**, of *f* and *g* is defined as

$$(f\circ g)(x)=f(g(x))$$

where the domain of $f\circ g$ is the set of all numbers *x* in the domain of *g* such that $g(x)$ is in the domain of *f*.

15. Let $f(x)=1+\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x}$. Find the following:

a) $(f+g)(x)$ and its domain.

b) $(f-g)(x)$ and its domain.

c) $(fg)(x)$ and its domain.

d) $\left(\dfrac{f}{g}\right)(x)$ and its domain.

e) $(f+g)(3)$

f) $(fg)(2)$

g) $(f\circ g)(x)$ and its domain.

h) $(g\circ f)(x)$ and its domain.

16. Let $f(x)=\sqrt{x-2}$ and $g(x)=\dfrac{2}{x}$. Find the following:

a) $(f\circ g)(x)$ and its domain.

b) $(g\circ f)(x)$ and its domain.