Section 12.4
Mathematical Induction

There are two aspects to mathematics – discovery and proof – and both are of equal importance. It is necessary to discover something before attempting to prove it, and we can only be certain of its truth once it has been proved. In this section we examine the relationship between these two parts of mathematics more closely.

Conjecture and Proof

1) Let's try a simple experiment. We add up more and more of the odd numbers as follows:

Sum of the first *n* odd numbers:

n=1 ________________________________ The sum of the first 1 odd number is

n=2 ________________________________ The sum of the first 2 odd numbers is

n=3 ________________________________ The sum of the first 3 odd numbers is

n=4 ________________________________ The sum of the first 4 odd numbers is

n=5 ________________________________ The sum of the first 5 odd numbers is

n=6 ________________________________ The sum of the first 6 odd numbers is

This leads naturally to the following question: Is it true that for every natural number *n*, the sum of the first *n* odd numbers is _____? Could this be true? We could try a few more numbers and find that the pattern persists for the first 6,7,8,9, and 10 odd numbers.

At this point, we feel quite sure that this is always true, so we make a **conjecture:**

The sum of the first n odd number is ________.

____

It is important to realize that this is still a *conjecture*. (Note that "conjecture" means an opinion or judgment that is not based on proof; a guess.) We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many).

2) Let $f(x) = x^2 + x + 41$. Find the values of $f(x)$ at 0, 1, 2, 3, 4, and 5.

$f(0) =$ ___________ $f(3) =$ ___________

$f(1) =$ ___________ $f(4) =$ ___________

$f(2) =$ ___________ $f(5) =$ ___________

This leads to the following question: Is it true that $f(x)$ is ____________________for any natural number *x*?

We could try a few more values: $f(6), f(7), f(8)$ and see that the pattern persists.

We feel quite sure to make a **conjecture:**

$f(x) = x^2 + x + 41$ is ____________________ for any *x* natural number.

It is important to realize that this is still a *conjecture*. We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many).

In this lies the power of mathematical proof. A **proof** is a clear argument that demonstrates the truth of a statement beyond doubt. We consider here a special kind of proof called mathematical induction that will help us prove statement like the one in (1) we have just considered.

However, the conjecture in (2) turns out to be false. Look at $f(40)$

Actually, $x = 40$ is the first natural number for which $f(x)$ is not prime.

Mathematical Induction

We consider a special kind of proof called mathematical induction. Here is how it works: Suppose we have a statement that says something about all natural numbers n. Let's call this statement *P*. For example, we could consider the statement

P: For every natural number *n*, the sum of the first *n* odd numbers is n^2.

Since this statement is about all natural numbers, it contains infinitely many statements; we will call them $P(1), P(2,)...$

$P(1)$: The sum of the first 1 odd number is 1^2.

$P(2)$: ____________________________________

$P(3)$: ____________________________________

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How can we prove all of these statements at once? Mathematical induction is a way of doing that. Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true.
In other words,

For every k, if $P(k)$ is true, then $P(k+1)$ is also true.

This is called the **induction step** because it leads us from the truth of one statement to the next.
Now, suppose that we can also prove that

$P(1)$ is true

The induction step now leads us through the following chain of statements:

$P(1)$ is true, so $P(2)$ is true.

$P(2)$ is true, so $P(3)$ is true.

$P(3)$ is true, so $P(4)$ is true.

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So we see that if both statements are proved, then statement *P* is proved for all n. We summarize this method of proof.

Principle of Mathematical Induction

For each natural number n, let $P(n)$ be a statement depending on n.
Suppose that the following two conditions are satisfied:

1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all natural numbers n.

To apply the principle, there are two steps:

Step 1 Prove that $P(1)$ is true.

Step 2 Assume that $P(k)$ is true and use this assumption (induction hypothesis) to prove that $P(k+1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that *if* $P(k)$ is true, *then* $P(k+1)$ is also true.

Example1 Prove that the conjecture we made in (1) is true.

Example 2 Prove that $4n < 2^n$ for all $n \geq 5$.

Example 3 Prove that $n^3 - n$ is divisible by 3 for all natural numbers n.

Example 4 Find $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n(n+1)}$