Section 11.5
Partial Fraction Decomposition

Let $f(x)=\dfrac{P(x)}{Q(x)}$ be a rational function where the degree of the numerator is less than the degree of the denominator (proper fraction).

Theorem Any polynomial with real coefficients can be factored completely into linear and irreducible quadratic factors, that is, factors of the form $ax+b$ and ax^2+bx+c, where a, b, and c are real numbers.

Definition A quadratic polynomials with no real zeros is called **irreducible** over the real numbers.

After we have completely factored the denominator Q, we can express $f(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

where i and j are natural numbers, and ax^2+bx+c is irreducible.
This sum is called the **partial fraction decomposition** of f.

We assume the proper rational expression $\dfrac{P(x)}{Q(x)}$ is in lowest terms. Any improper rational expression can be reduced by long division to a mixed form consisting of the sum of a polynomial and a proper rational expression.

There are four cases.

Distinct Linear Factors

Case 1: The denominator is a product of distinct (nonrepeated) linear factors

Suppose that we can factor $Q(x)$ as

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$$

with no factor repeated. In this case the partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$ takes the form

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\dots+\frac{A_n}{a_nx+b_n}$$

Example: $$\frac{2x+7}{(x+1)(x-2)(2x+3)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{2x+3}$$

Repeated Linear Factors

Case 2: The denominator is a product of linear factors, some of which are repeated

Suppose the complete factorization of $Q(x)$ contains the linear factors $ax+b$ repeated k times; that is, $(ax+b)^k$ is a factor of $Q(x)$. Then, corresponding to each such factor, the partial fraction decomposition for $\frac{P(x)}{Q(x)}$ contains

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$$

Example: $$\frac{x^2+1}{x(x+1)(x-2)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}+\frac{D}{(x-2)^2}+\frac{E}{(x-2)^3}$$

Irreducible Quadratic Factors

Case 3: The denominator has irreducible quadratic factors, none of which is repeated

Suppose the complete factorization of $Q(x)$ contains the quadratic factor ax^2+bx+c which cannot be factored further. Then, corresponding to this, the partial fraction decomposition for $\frac{P(x)}{Q(x)}$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Example: $$\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

Repeated Irreducible Quadratic Factors

Case 4: The denominator has a repeated irreducible quadratic factor

Suppose the complete factorization of $Q(x)$ contains the factor $(ax^2+bx+c)^k$ where ax^2+bx+c cannot be factored further. Then the partial fraction decomposition for $\frac{P(x)}{Q(x)}$ will have the terms

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Example: $$\frac{x^5-3x^2+12x-1}{x^3(x^2+x+1)(x^2+2)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+x+1}+\frac{Fx+G}{x^2+2}+\frac{Hx+I}{(x^2+2)^2}+\frac{Jx+K}{(x^2+2)^3}$$