5.2 Solutions by Matrices

Solving Linear Systems using Matrices

Definition: **A MATRIX** is a rectangular array of numbers or entries (elements).

Matrix – Matrices (plural)

"Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed.

The **beginnings** of matrices and determinants go back to the **second century BC**. However it was not until near the end of the 17th Century that the ideas reappeared and development really got underway. It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. **The Babylonians** studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive. For example a tablet dating from around 300 BC contains the following problem:

There are two fields whose total area is 1800 *square yards. One produces grain at the rate of* $^2/_3$ *of a bushel per square yard while the other produces grain at the rate of* $^1/_2$ *a bushel per square yard. If the total yield is* 1100 *bushels, what is the size of ach field.*

Write a system of two equations with two variables that models the Babylonian problem. **Can You solve it ?**

- STEP 1 – Represent each unknown by a separate variable

- STEP 2 - Write the conditions stated in the problem as two equations

- STEP 3 – Solve the system.

The Chinese, between 200 BC and 100 BC, came much closer to matrices than the Babylonians. Indeed it is fair to say that the text Nine Chapters on the Mathematical Art written during the Han Dynasty gives the **first known example of matrix methods.** First a problem is set up which is similar to the Babylonian example:

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 *measures. Two of the first, three of the second and one of the third make* 34 *measures. And one of the first, two of the second and three of the third make* 26 *measures. How many measures of corn are contained of one bundle of each type?*

Write a system of three equations with three variables that models the Chinese problem.

- STEP 1 – Represent each unknown by a separate variable

- STEP 2 - Write the conditions stated in the problem as three equations

Now the author does something quite remarkable. He sets up the coefficients of the system of three linear equations in three unknowns as a table on a 'counting board'.

3	2	1	39
2	3	1	34
1	2	3	26

Most remarkably the author, writing in 200 BC, instructs the reader how to solve the system by the matrix method.

This method, now known as **Gaussian elimination**, would not become well known until the early 19^{th} Century.

THE COEFFICIENT MATRIX

- the entries are the coefficients of the variables

THE AUGMENTED MATRIX

- each row represents one equation of the system

1^{st} equation________________________________

2^{nd} equation________________________________

3^{rd} equation________________________________

EXAMPLES OF MATRICES

DIMENSION OF A MATRIX

<u>**Exercise #1**</u> **What is the augmented matrix for each of the following systems?**

a) $\begin{cases} x-2y-2z=4 \\ 2x+y-3z=\dfrac{7}{2} \\ x-y-z=3 \end{cases}$

b) $\begin{cases} 3x-z=7 \\ 2x+y=6 \\ 3y-z=7 \end{cases}$

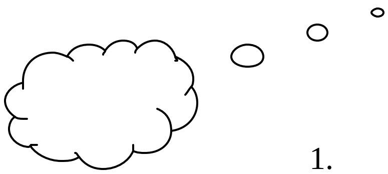
<u>**Exercise #2**</u> **Solve the following system using back-substitution:**

$$\begin{cases} x-3y+2z=5 \\ \quad 2y-z=4 \\ \quad\quad 4z=8 \end{cases}$$

Write its augmented matrix. What are the entries in the left corner (below the diagonal)?

This matrix is written in ___________________________

Given a system of linear equations, using matrix representation, how can we can obtain an equivalent matrix in upper triangular form?



How can we obtain equivalent equations?

What operations can we perform on the equations of a system?

1. ______________________________

2. ______________________________

3. ______________________________

ELEMENTARY ROW OPERATIONS

1. ______________________________
2. ______________________________
3. ______________________________

<u>Exercise #3</u> **Perform the given elementary row operations on the following matrices:**

a) Multiply row 2 by -3:

$$\begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}$$

b) Multiply row 1 by ¼

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 5 & 4 \end{bmatrix}$$

c) Interchange row 1 and row 3:

$$\begin{bmatrix} 0 & -3 & 2 & -3 \\ 2 & 6 & -1 & 3 \\ 1 & 0 & -2 & 5 \end{bmatrix}$$

d) Add 2 (row 1) to row 2:

$$\begin{bmatrix} 1 & -3 & 6 \\ -2 & 4 & -1 \end{bmatrix}$$

e) Add -4(row 1) to row 3:

$$\begin{bmatrix} 1 & 2 & 1 & -5 \\ 0 & 4 & -2 & 3 \\ 4 & -1 & 6 & -8 \end{bmatrix}$$

f) Add 2(row 2) to row 3:

$$\begin{bmatrix} 1 & -7 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & -2 & -3 & 4 \end{bmatrix}$$

<u>Exercise #4</u> **Use row operations to obtain an equivalent matrix in upper triangular form:**

$$\begin{bmatrix} 2 & -6 & 2 & -8 \\ 3 & -1 & -1 & 8 \\ 2 & -2 & 3 & -1 \end{bmatrix}$$

STEP 1 – Make the first entry of the first row equal to 1 by ___________________________________

STEP 2 – Obtain zeros in the lower two entries of the first column .

Obtain zero on the 1st entry of the second row by_________________________________

Obtain zero on the 1st entry of the third row by____________________________________

STEP 3 – Obtain a zero as the second entry of the third row by ________________________________

<u>Exercise #5</u> **Use row operations to obtain an equivalent matrix in upper triangular form:**

$$\begin{bmatrix} 1 & -2 & 4 & 3 \\ 5 & -7 & 8 & 6 \\ -2 & 6 & -7 & 6 \end{bmatrix}$$

Matrices have wide **applications** in mathematics, business, science, and engineering. Olga Taussky-Todd (1906-1995) was one of the world's leaders in developing applications of Matrix Theory. She successfully applied matrices to the study of aerodynamics, a field used in the design of airplanes and rockets. She was for many years a professor of mathematics at Caltech in Pasadena.

Exercise #6 **Use matrix reduction (Gaussian elimination) to solve the system:** $\begin{cases} x+3y=11 \\ 2x-y=1 \end{cases}$

Exercise #7 **Use matrix reduction (Gaussian elimination) to solve the system:** $\begin{cases} 2x-4y=6 \\ 3x-4y+z=8 \\ 2x-3z=-11 \end{cases}$

Exercise #8 **Use matrix reduction (Gaussian elimination) to solve the system:** $\begin{cases} 2x - y = 6 \\ 4x - 2y = 0 \end{cases}$

Exercise #9 **Use matrix reduction to solve the system:** $\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$

<u>Exercise #</u>10 Use matrix reduction to solve the system:

$$\begin{cases} x+3y-2z-w=9 \\ 4x+y+z+2w=2 \\ -3x-y+z-w=-5 \\ x-y-3z-2w=2 \end{cases}$$

5.7 The Algebra of Matrices

Up to this point we've been using matrices simply as a notational convenience. Matrices have many other uses in mathematics and sciences. These applications include: electronics (finding the currents in a circuit), engineering (finding the forces in a bridge or truss), genetics (working out selection processes), chemistry (finding quantities in a chemical solution), economics (study of stock market trends, optimization of profit and minimization of loss), describing the quantum mechanics of atomic structure, designing computer game graphics, etc. For most of these applications a knowledge of matrix algebra is essential. Like numbers, matrices can be added, subtracted, multiplied, and divided.

Notations

- It is often convenient to define a single symbol to represent the entire matrix. Conventionally this will be an upper case letter, e.g. A.
- The **elements in a matrix *A*** are denoted by a_{ij}, where i is the row number and j is the column number.

Example: In the matrix

A= $\begin{bmatrix} 1 & -2 & 4 & 3 \\ 5 & -7 & 8 & 6 \\ -2 & 6 & -7 & 6 \end{bmatrix}$, the element $a_{13} = 4$, since the element in the 1st row and 3rd column is 4.

List the following elements of A: $a_{22} =$ ______ $a_{24} =$ ______ $a_{31} =$ ______

TWO MATRICES ARE EQUAL
if they have the same size and the corresponding entries are equal.

Example: If $\begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & a \end{bmatrix}$ then ____________________

THE IDENTITY MATRIX, called **I**,
is a square matrix with all elements 0 except the principal diagonal which has all ones.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2 x 2 identity matrix; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3 x 3 identity matrix.

SUM, DIFFERENCE, AND SCALAR PRODUCT OF MATRICES

If *A* and *B* are matrices of the same dimension and if k is any real number, then:

1. The **sum** *A* + *B* is the matrix of the same dimension as *A* and *B*, and its (*i*,*j*) entry is $a_{ij} + b_{ij}$.
2. The **difference** *A* –*B* is the matrix of the same dimension as *A* and *B*, and its (*i*,*j*) entry is $a_{ij} - b_{ij}$.
3. The **scalar product** *kA* is the matrix of the same dimension as *A*, and its (*i*,*j*) entry is ka_{ij}.

Exercise #1 **Performing Algebraic Operations on Matrices**

Carry out each indicated operation, or explain why it cannot be performed:

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

a) $A + B =$

b) $C - D =$

c) $C + A =$

d) $5A =$

MATRIX MULTIPLICATION

The **product *AB*** of two matrices *A* and *B* is **defined** only when the **number of columns in *A* is equal to the number of rows in *B*.** This means that when we write their dimensions side by side, their inner numbers must match:

Matrices: *A* *B*

Dimensions: m x n n x k

columns in A rows in B

Then *AB* will have dimension m x k.

What must be true about the dimensions of the matrices *A* and *B* if both products *AB* and *BA* are defined?

INNER PRODUCT OF A ROW OF *A* AND A COLUMN OF *B*

If $\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$ is a row of *A*, and if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a column of *B*, then their **inner product** is the number

$a_1b_1 + a_2b_2 + \ldots + a_nb_n$.

Calculate: $\begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix} =$

THE PRODUCT OF TWO MATRICES

Suppose that A is an $m\ x\ n$ matrix and B an $n\ x\ k$ matrix. Then $C = AB$ is an $m\ x\ k$ matrix, where c_{ij} is the inner product of the ith row and the jth column of B.

<u>**Exercise #2**</u> **Calculate, if possible, the products AB and BA,** where $A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$.

MATRIX MULTIPLICATION IS NOT COMMUTATIVE!

If $A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$, calculate AB and BA.

<u>Exercise #3</u> Representing Demographic Data in Terms of Matrices

In a certain city the proportion of voters in each age group who are registered as Democrats, Republicans or Independents is given by the following matrix:

	Age		
	18-30	31-50	over 50
Democrat	0.30	0.60	0.50
Republican	0.50	0.35	0.25
Independent	0.20	0.05	0.25

$$\begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} = A$$

The next matrix gives the distribution, by age and sex, of the voting population of this city.

		Male	Female
	18 – 30	5,000	6,000
Age	31 – 50	10,000	12,000
	over 50	12,000	15,000

$$\begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = B$$

For the purpose of this problem, let's make the (highly unrealistic) assumption that within each age group, political preference is not related to gender. That is, the percentage of Democrat male in the 18 – 30 group, for example, is the same as the percentage of Democrat females in this group.

a) Calculate the product AB.
b) How many males are registered as Democrats in this city?
c) How many females are registered as Republicans?

(Hint: When we take the inner product of a row from A with a column from B, we are adding the number of people in each of the three age groups who belong to the category in question. For example, the (2,1) entry of AB (the 9,000) was obtained by taking the inner product of the Republican row from A with the Male column from B. This number is therefore the total number of male Republicans in this city.)

Exercise #4 A small fast-food chain has restaurants in Santa Monica, Long Beach, and Anaheim. Only hamburgers, hot dogs, and milk shakes are sold by this chain. On a certain day, sales were distributed according to the following matrix:

Number of items sold

SM =Santa Monica, LB =Long Beach, A = Anaheim

$$\begin{array}{r} \\ \text{Hamburgers} \\ \text{Hot dogs} \\ \text{Milk shakes} \end{array} \begin{array}{c} \begin{array}{ccc} \text{SM} & \text{LB} & \text{A} \end{array} \\ \begin{bmatrix} 4000 & 1000 & 3500 \\ 400 & 300 & 200 \\ 700 & 500 & 9000 \end{bmatrix} = A \end{array}$$

The price of each item is given by the following matrix:

Hambg. Hot Dog Milk Shake

$$\begin{bmatrix} \$0.90 & \$0.80 & \$1.10 \end{bmatrix}$$

a) Calculate the product BA.
b) Interpret the entries in the product matrix BA.

Exercise #5 Let O represent the 2 x 2 **zero matrix**: $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

a) If A and B are 2x 2 matrices with AB =0, is it necessarily true that A =0 or B=0 ? Justify your answer.

b) Find a matrix $A \neq 0$ such that $A^2 = 0$.

Exercise #6 a) If A and B are 2 x2 matrices, is it necessarily true that $(A+B)^2 = A^2 + 2AB + B^2$? Justify your answer.

b) What is, in general, $(A+B)^2$?

Exercise #7 Let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.

Exercise #8 Let $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.