5.2 Solutions by Matrices

Solving Linear Systems using Matrices

Definition: **A MATRIX** is a rectangular array of numbers or entries (elements).

Matrix – Matrices (plural)

"Matrix" is the Latin word for womb, and it retains that sense in English.
It can also mean more generally is any place in which something is formed

The **beginnings** of matrices and determinants goes back to the **second century BC**. However it was not until near the end of the 17th Century that the ideas reappeared and development really got underway. It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. **The Babylonians** studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive. For example a tablet dating from around 300 BC contains the following problem:

There are two fields whose total area is 1800 *square yards. One produces grain at the rate of $^2/_3$ of a bushel per square yard while the other produces grain at the rate of $^1/_2$ a bushel per square yard. If the total yield is* 1100 *bushels, what is the size of ach field.*

Write a system of two equations with two variables that models the Babylonian problem. **Can You solve it ?**

- STEP 1 – Represent each unknown by a separate variable

let x = size of first field (the number of square yards)
y = size of second field

- STEP 2 - Write the conditions stated in the problem as two equations

total # of bushels $\left\{ \frac{2}{3}x + \frac{1}{2}y = 1100 \right.$ $\quad | \cdot 6$
total # of fields $\quad x + y = 1800$

- STEP 3 – Solve the system.

$$\begin{cases} 4x + 3y = 6600 \\ x + y = 1800 \end{cases}$$

$x = 1200$ sq. yards
$y = 600$ sq. yards

The Chinese, between 200 BC and 100 BC, came much closer to matrices than the Babylonians. Indeed it is fair to say that the text Nine Chapters on the Mathematical Art written during the Han Dynasty gives the **first known example of matrix methods.** First a problem is set up which is similar to the Babylonian example:

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained of one bundle of each type?

Write a system of three equations with three variables that models the Chinese problem.

- STEP 1 – Represent each unknown by a separate variable

let x = # measures of 1st type/bundle
y = # measures of 2nd type/bundle
z = # measures of 3rd type/bundle

- STEP 2 - Write the conditions stated in the problem as three equations

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases}$$

Now the author does something quite remarkable. He sets up the coefficients of the system of three linear equations in three unknowns as a table on a 'counting board'.

3	2	1	39
2	3	1	34
1	2	3	26

Most remarkably the author, writing in 200 BC, instructs the reader how to solve the system by the matrix method.

This method, now known as **Gaussian elimination**, would not become well known until the early 19th Century.

Friedrich Gauss (1777-1855)

THE COEFFICIENT MATRIX
- the entries are the coefficients of the variables

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

THE AUGMENTED MATRIX
- each row represents one equation of the system

1st equation $3x + 2y + z = 39$

2nd equation $2x + 3y + z = 34$

3rd equation $x + 2y + 3z = 26$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 30 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right]$$

EXAMPLES OF MATRICES

$\begin{bmatrix} 1 & 2 & 5 \\ -3 & 4 & -7 \end{bmatrix}$ 2 x 3 matrix

$[4]$ 1x1 matrix

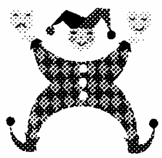
$\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ 3 x 1 matrix

$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -5 \\ 11 & 0 & 15 \end{bmatrix}$ 3x 3 matrix

DIMENSION OF A MATRIX $n \times m$

n: # of rows

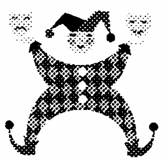
m: # of columns



What is the augmented matrix for each of the following systems?

a) $\begin{cases} x-2y-2z=4 \\ 2x+y-3z=\dfrac{7}{2} \\ x-y-z=3 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & -2 & -2 & 4 \\ 2 & 1 & -3 & \frac{7}{2} \\ 1 & -1 & -1 & 3 \end{array}\right]$

b) $\begin{cases} 3x-z=7 \\ 2x+y=6 \\ 3y-z=7 \end{cases}$ $\left[\begin{array}{ccc|c} 3 & 0 & -1 & 7 \\ 2 & 1 & 0 & 6 \\ 0 & 3 & -1 & 7 \end{array}\right]$



Solve the following system using back-substitution:

$\begin{array}{l} (1^\circ) \\ (2^\circ) \\ (3^\circ) \end{array} \begin{cases} x-3y+2z=5 \\ 2y-z=4 \\ 4z=8 \end{cases}$

$(3^\circ) \Rightarrow z = 2$

$(2^\circ) \Rightarrow 2y - 2 = 4 \Rightarrow y = 3$

$(1^\circ) \Rightarrow x - 3(3) + 2(2) = 5 \Rightarrow x = 10$

The solution is $(10, 3, 2)$

Write its augmented matrix. What are the entries in the left corner (below the diagonal)?

$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$

This is an upper triangular form

So, given a system
using matrix representation
obtain an equivalent matrix in upper triangular form

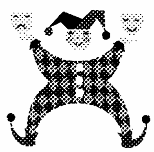
How can we obtain equivalent equations?

What operations can we perform on the equations of a system?

1. Multiply / Divide both sides of the equation by $k \neq 0$
2. Add / Subtract a constant multiple of one equation to another equation
3. Interchange two equations.

ELEMENTARY ROW OPERATIONS

1. $\cdot$ or $\div$ the entries of any row by $k \neq 0$
2. $+$ or $-$ a constant multiple of one row to another row
3. interchange two rows.



Perform the given elementary row operations on the following matrices:

a) Multiply row 2 by -3:

$$\begin{array}{c} \\ -3 \end{array}\begin{bmatrix}-2 & 1 & 0\\ 3 & -1 & 2\end{bmatrix}\xrightarrow{R_2 \to (-3)R_2}\begin{bmatrix}-2 & 1 & 0\\ -9 & 3 & -6\end{bmatrix}$$

b) Multiply row 1 by ¼

$$\begin{array}{c} \frac{1}{4} \\ \\ \end{array}\begin{bmatrix}2 & 0 & 3\\ -1 & 5 & 4\end{bmatrix}\xrightarrow{R_1 \to \frac{1}{4}R_1}\begin{bmatrix}\frac{1}{2} & 0 & \frac{3}{4}\\ -1 & 5 & 4\end{bmatrix}$$

c) Interchange row 1 and row 3:

$$\begin{bmatrix}0 & -3 & 2 & -3\\ 2 & 6 & -1 & 3\\ 1 & 0 & -2 & 5\end{bmatrix}\xrightarrow{R_1 \leftrightarrow R_3}\begin{bmatrix}1 & 0 & -2 & 5\\ 2 & 6 & -1 & 3\\ 0 & -3 & 2 & -3\end{bmatrix}$$

d) Add 2 (row 1) to row 2:

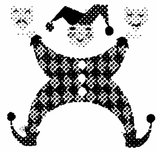
$$\begin{array}{c} 2 \\ \\ \end{array}\begin{bmatrix}1 & -3 & 6\\ -2 & 4 & -1\end{bmatrix}\xrightarrow{R_2 \to R_2 + 2R_1}\begin{bmatrix}1 & -3 & 6\\ 0 & -2 & 11\end{bmatrix}$$

e) Add -4(row 1) to row 3:

$$\begin{array}{c} -4 \\ \\ \\ \end{array}\begin{bmatrix}1 & 2 & 1 & -5\\ 0 & 4 & -2 & 3\\ 4 & -1 & 6 & -8\end{bmatrix}\xrightarrow{R_3 \to R_3 - 4R_1}\begin{bmatrix}1 & 2 & 1 & -5\\ 0 & 4 & -2 & 3\\ 0 & -9 & 2 & 12\end{bmatrix}$$

f) Add 2(row 2) to row 3:

$$\begin{array}{c} \\ 2 \\ \\ \end{array}\begin{bmatrix}1 & -7 & 5 & 2\\ 0 & 1 & -3 & -1\\ 0 & -2 & -3 & 4\end{bmatrix}\xrightarrow{R_3 \to R_3 + 2R_2}\begin{bmatrix}1 & -7 & 5 & 2\\ 0 & 1 & -3 & -1\\ 0 & 0 & -9 & 2\end{bmatrix}$$



Use row operations to obtain an equivalent matrix in upper triangular form:

$$\frac{1}{2}\begin{bmatrix} 2 & -6 & 2 & -8 \\ 3 & -1 & -1 & 8 \\ 2 & -2 & 3 & -1 \end{bmatrix}$$

STEP 1 – Make the first entry of the first row equal to 1 by $\div R_1$ by 2 $\qquad \frac{1}{2}R_1$

$$\begin{pmatrix} 1 & -3 & 1 & -4 \\ \boxed{3} & -1 & -1 & 8 \\ \boxed{2} & -2 & 3 & -1 \end{pmatrix} -3$$

STEP 2 – Obtain zeros in the lower two entries of the first column .

Obtain zero on the 1st entry of the second row by $R_2 - 3R_1$

$$\begin{pmatrix} 1 & -3 & 1 & -4 \\ 0 & 8 & -4 & 20 \\ \boxed{2} & -2 & 3 & -1 \end{pmatrix} -2$$

Obtain zero on the 1st entry of the third row by $R_3 - 2R_1$

$$\begin{pmatrix} 1 & -3 & 1 & -4 \\ 0 & 8 & -4 & 20 \\ 0 & \boxed{4} & 1 & 7 \end{pmatrix} -\frac{1}{2}$$

STEP 3 – Obtain a zero as the second entry of the third row by $R_3 - \frac{1}{2}R_2$

$$\begin{pmatrix} 1 & -3 & 1 & -4 \\ 0 & 8 & -4 & 20 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

Use row operations to obtain an equivalent matrix in upper triangular form:

$$\begin{bmatrix} 1 & -2 & 4 & 3 \\ \boxed{5} & -7 & 8 & 6 \\ -2 & 6 & -7 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{pmatrix} 1 & -2 & 4 & 3 \\ 0 & \boxed{3} & -12 & -9 \\ -2 & 6 & -7 & 6 \end{pmatrix} \div 3 \xrightarrow{R_2 \to \frac{1}{3}R_2,}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & -4 & -3 \\ \boxed{-2} & 6 & -7 & 6 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_1}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & \boxed{2} & 1 & 12 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 9 & 18 \end{pmatrix}$$

Matrices have wide **applications** in mathematics, business, science, and engineering. Olga Taussky-Todd (1906-1995) was one of the world's leaders in developing applications of Matrix Theory. She successfully applied matrices to the study of aerodynamics, a field used in the design of airplanes and rockets. She was for many years a professor of mathematics at Caltech in Pasadena.

Use matrix reduction (Gaussian elimination **) to solve the system:** $\begin{cases} x+3y=11 \\ 2x-y=1 \end{cases}$

$$-2 \quad \left(\begin{array}{cc|c} 1 & 3 & 11 \\ \boxed{2} & -1 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 3 & 11 \\ 0 & -7 & -21 \end{array}\right)$$

2nd row: $-7y = -21 \Rightarrow y = 3$

1st row: $x + 3y = 11$

$x + 3(3) = 11 \Rightarrow x = 2$

The solution is $(2,3)$.

Use matrix reduction (Gaussian elimination **) to solve the system:** $\begin{cases} 2x-4y=6 \\ 3x-4y+z=8 \\ 2x-3z=-11 \end{cases}$

$$\div 2 \quad \left(\begin{array}{ccc|c} \boxed{2} & -4 & 0 & 6 \\ 3 & -4 & 1 & 8 \\ 2 & 0 & -3 & -11 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{cccc} 1 & -2 & 0 & 3 \\ \boxed{3} & -4 & 1 & 8 \\ 2 & 0 & -3 & -11 \end{array}\right) -3 \longrightarrow$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{cccc} 1 & -2 & 0 & 3 \\ 0 & 2 & 1 & -1 \\ \boxed{2} & 0 & -3 & -11 \end{array}\right) -2 \xrightarrow{R_3 \to R_3 - 2R_1}$$

$$\longrightarrow \left(\begin{array}{cccc} 1 & -2 & 0 & 3 \\ 0 & 2 & 1 & -1 \\ 0 & \boxed{4} & -3 & -17 \end{array}\right) -2 \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -5 & -15 \end{array}\right)$$

3rd row: $-5z = -15 \Rightarrow z = 3$

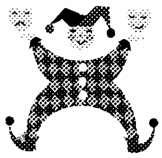
2nd row: $2y + z = -1$

$2y + 3 = -1 \Rightarrow y = -2$

1st row: $x - 2y = 3$

$x - 2(-2) = 3 \Rightarrow x = -1$

The solution is $(-1,-2,3)$



Gaussian elimination

Use matrix reduction (Gaussian elimination) to solve the system: $\begin{cases} 2x - y = 6 \\ 4x - 2y = 0 \end{cases}$

$$\begin{array}{c} \div 2 \\ \div 2 \end{array}\left(\begin{array}{cc|c} \boxed{2} & -1 & 6 \\ 4 & -2 & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{ccc} 1 & -\frac{1}{2} & 3 \\ 2 & -1 & 0 \end{array}\right)^{-2} \xrightarrow{R_2 \to R_2 - 2R_1}$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & -6 \end{array}\right)$$

2nd row: $0 = -6$ Contradiction

$\Rightarrow$ The system has no solutions

Use matrix reduction (Gauss – Jordan method) to solve the system: $\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$

$$\left(\begin{array}{ccc|c} \boxed{2} & -5 & 3 & 1 \\ 1 & -2 & -2 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc} 1 & -2 & -2 & 8 \\ \boxed{2} & -5 & 3 & 1 \end{array}\right)^{-2} \longrightarrow$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & \boxed{-2} & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right)_{-2} \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & -1 & 7 & -15 \end{array}\right)$$

2nd row: $-y + 7z = -15$

1st row: $x - 16z = 38$

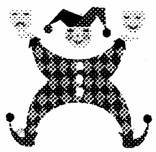
Solve the equations for x and y, respectively.

$\boxed{y = 7z + 15}$

$\boxed{x = 16z + 38}$

The solution set: $\{(16z + 38,\ 7z + 15,\ z)\}$

There is an infinite number of solutions.



Gaussian elimination OR

Use matrix reduction (Gauss – Jordan method) to solve the system:

$$\begin{cases} x+3y-2z-w=9 \\ 4x+y+z+2w=2 \\ -3x-y+z-w=-5 \\ x-y-3z-2w=2 \end{cases}$$

$-1 \quad 3 \quad -4$

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ \boxed{4} & 1 & 1 & 2 & 2 \\ \boxed{-3} & -1 & 1 & -1 & -5 \\ \boxed{1} & -1 & -3 & -2 & 2 \end{array}\right) \xrightarrow{\substack{R_2 \to R_2-4R_1 \\ R_3 \to R_3+3R_1 \\ R_4 \to R_4-R_1}} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 0 & \boxed{-11} & 9 & 6 & -34 \\ 0 & \boxed{8} & -5 & -4 & 22 \\ 0 & -4 & -1 & -1 & -7 \end{array}\right) \begin{array}{c} \\ \div 11 \\ \\ 2 \end{array}$$

$$\xrightarrow{\substack{R_2 \to \frac{1}{11}R_2 \\ R_3 \to R_3+2R_4}} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 0 & -1 & \frac{9}{11} & \frac{6}{11} & \frac{-34}{11} \\ 0 & 0 & -7 & -6 & +8 \\ 0 & -4 & -1 & -1 & -7 \end{array}\right) \begin{array}{c} \\ -4 \\ \\ \\ \end{array} \xrightarrow{R_4 \to R_4-4R_2} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 0 & -1 & \frac{9}{11} & \frac{6}{11} & \frac{-34}{11} \\ 0 & 0 & \boxed{-7} & -6 & +8 \\ 0 & 0 & \frac{-47}{11} & \frac{-35}{11} & \frac{59}{11} \end{array}\right) \begin{array}{c} \\ \\ \div 11 \\ \\ \end{array}$$

$$\xrightarrow{R_3 \to \frac{1}{11}R_3} \left(\begin{array}{ccccc} 1 & 3 & -2 & -1 & 9 \\ 0 & -1 & \frac{9}{11} & \frac{6}{11} & \frac{-34}{11} \\ 0 & 0 & -\frac{7}{11} & \frac{-6}{11} & \frac{+8}{11} \\ 0 & 0 & \boxed{\frac{-47}{11}} & \frac{-35}{11} & \frac{59}{11} \end{array}\right) \begin{array}{c} \\ \\ \frac{-47}{7} \\ \\ \end{array} \xrightarrow{R_4 \to R_4-\frac{47}{11}R_3} \left(\begin{array}{ccccc} 1 & 3 & -2 & -1 & 9 \\ 0 & -1 & \frac{9}{11} & \frac{6}{11} & \frac{-34}{11} \\ 0 & 0 & -\frac{7}{11} & \frac{-6}{11} & \frac{+8}{11} \\ 0 & 0 & 0 & \frac{37}{77} & \frac{37}{77} \end{array}\right) \begin{array}{c} \\ \cdot 11 \\ \cdot 11 \\ \cdot 77 \end{array}$$

$$\xrightarrow{\substack{R_2 \to 11R_2 \\ R_3 \to 11R_3 \\ R_4 \to 77R_4}} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 0 & -11 & 9 & 6 & -34 \\ 0 & 0 & -7 & -6 & 8 \\ 0 & 0 & 0 & 37 & 37 \end{array}\right)$$

4th row: $37w = 37 \Rightarrow w = 1$

3rd row: $-7z - 6w = 8$

$-7z - 6 = 8 \Rightarrow z = -2$

2nd row: $-11y + 9z + 6w = -34$

$-11y - 18 + 6 = -34$

$-11y = -22 \Rightarrow y = 2$

1st row: $x + 3y - 2z - w = 9$

$x + 6 + 4 - 1 = 9 \quad x = 0$

The solution set is $\boxed{(0, 2, -2, 1)}$