Math 130

Handout 4.5 & 4.6

SOLUTIONS - SELECTED EXERCISES

6e

$$10^{x+3} = 5e^{7-x} \quad / \ln$$

$$\ln 10^{x+3} = \ln(5e^{7-x})$$

$$(x+3)\ln 10 = \ln 5 + \ln e^{7-x}$$

$$x\ln 10 + 3\ln 10 = \ln 5 + 7 - x$$

$$x\ln 10 + x = \ln 5 + 7 - 3\ln 10$$

$$x(1+\ln 10) = \ln 5 + 7 - 3\ln 10$$

$$\boxed{x = \frac{\ln 5 - 3\ln 10 + 7}{1+\ln 10}}$$

or

$$x = \frac{\ln\frac{5}{10^3} + 7}{1+\ln 10} \approx 0.515$$

6j

$$5^x = 3^{2x-1} \quad / \ln$$

$$\ln 5^x = \ln 3^{2x-1}$$

$$x\ln 5 = (2x-1)\ln 3$$

$$x\ln 5 = 2x\ln 3 - \ln 3$$

$$\ln 3 = 2x\ln 3 - x\ln 5$$

$$\ln 3 = x(2\ln 3 - \ln 5)$$

$$\boxed{x = \frac{\ln 3}{2\ln 3 - \ln 5}} \quad \text{or}$$

$$x = \frac{\ln 3}{\ln\frac{9}{5}} \approx 1.87$$

6l

$$\log_8(x+5) - \log_8 2 = 1$$

Condition: $x+5>0 \Rightarrow x>-5$

$$\log_8 \frac{x+5}{2} = 1$$

$$\Leftrightarrow$$

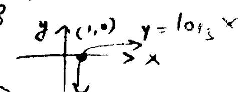
$$8^1 = \frac{x+5}{2}$$

$$x+5 = 16 \Rightarrow \boxed{x = 11}$$

6n

$$\log_2(\log_3 x) = -1$$

Conditions $\begin{cases} x>0 \\ \log_3 x > 0 \Leftrightarrow x>1 \end{cases}$



Therefore, $x>1$

$$\log_2(\log_3 x) = -1$$

$$\Leftrightarrow$$

$$2^{-1} = \log_3 x$$

$$\frac{1}{2} = \log_3 x$$

$$\Leftrightarrow$$

$$3^{\frac{1}{2}} = x$$

$$\boxed{x = \sqrt{3}} \quad > 1$$

6p

$$\ln(-x) + \ln 3 = \ln(2x-15)$$

Conditions $\begin{cases} -x>0 \\ \text{and} \\ 2x-15>0 \end{cases}$ $\begin{cases} x<0 \\ \text{and} \\ x>\frac{15}{2} \end{cases}$

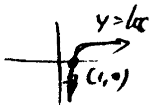
Therefore, $x \in \emptyset$

No solutions!

6s

$$\log x = \sqrt{\log x}$$

Conditions $\begin{cases} x>0 \text{ and} \\ \log x \geq 0 \iff x \geq 1 \end{cases}$



So. $x \geq 1$

let $\log x = t$

$t = \sqrt{t} \quad /^2$

$t^2 = t$

$t^2 - t = 0$

$t(t-1) = 0 \quad \begin{cases} t=0 \\ t=1 \end{cases}$

$t = 0 \Rightarrow$ OR $t = 1 \Rightarrow$

$\log x = 0$ $\qquad$ $\log x = 1$

$x = 1 \geq 1$ $\qquad$ $x = 10 \geq 10$

So $\boxed{x \in \{1, 10\}}$

7g

$ae^{kt} = e^{bt} \quad / \div e^{bt} \neq 0$

$a\dfrac{e^{kt}}{e^{bt}} = 1$

$ae^{kt-bt} = 1$

$e^{t(k-b)} = \dfrac{1}{a} \quad / \ln$

$\ln e^{t(k-b)} = \dfrac{1}{a}$

$t(k-b) = \dfrac{1}{a}$

$\boxed{t = \dfrac{1}{a(k-b)}}$ $\qquad k \neq b$

7h

$i = \dfrac{E}{R}\left(1 - e^{-\frac{Rt}{2}}\right)$ $\qquad t = ?$

$\dfrac{Ri}{E} = 1 - e^{-\frac{Rt}{2}}$

$e^{-\frac{Rt}{2}} = 1 - \dfrac{Ri}{E} \quad / \ln$

$\ln e^{-\frac{Rt}{2}} = \ln\left(1 - \dfrac{Ri}{E}\right)$

$-\dfrac{Rt}{2} = \ln\left(1 - \dfrac{Ri}{E}\right)$

$\boxed{t = \dfrac{-2}{R}\ln\left(1 - \dfrac{Ri}{E}\right)}$

8a

$P = 2e^{-0.5t} \Rightarrow P = P_0 a^t$

P_0 = initial value $(t=0)$

$t=0$, $P = P_0 = 2$

we want $e^{-0.5t} = a^t$

$\left(e^{-0.5}\right)^t = a^t$

$\Rightarrow a = e^{-0.5} \approx 0.61 = a$

so $\boxed{P = 2(0.61)^t}$

9b

$P = 4(0.55)^t \Rightarrow P = P_0 e^{kt}$

$P_0 = 4$

we want $(0.55)^t = e^{kt}$

$(0.55)^t = \left(e^k\right)^t$

$\Rightarrow e^k = 0.55 \quad / \ln$

$\ln e^k = \ln 0.55$

$k = \ln 0.55 \approx -0.6$

so $\boxed{P = 4e^{-0.6t}}$

10a

$f(t) = 50e^{0.1t}$

f is one-to-one (increasing on its entire domain), so it has an inverse

1st $y = 50e^{0.1t}$

and solve for t

$\frac{y}{50} = e^{0.1t} \quad / \ln$

$\ln\left(\frac{y}{50}\right) = \ln\left(e^{0.1t}\right)$

$0.1t = \ln \frac{y}{50}$

$t = \frac{1}{0.1} \ln \frac{y}{50}$

$t = 10 \ln\left(\frac{y}{50}\right)$

3rd $y \longleftrightarrow t$

$y = 10 \ln \frac{t}{50}$

$\boxed{f^{-1}(t) = 10 \ln\left(\frac{t}{50}\right)}$

(10b)

$f(t) = 1 + \ln t$

f is increasing on its entire domain $\Rightarrow$

f is one-to-one $\Rightarrow$

f has an inverse

1st $y = 1 + \ln t$

and solve for t

$\ln t = y - 1$

$e^{y-1} = t$

3rd $t \leftrightarrow y$

$e^{t-1} = y$

$\boxed{f^{-1}(t) = e^{t-1}}$

(11)

$P = P_0 e^{-kt}$, $k = ?$

t = time (hrs)

P = quantity of pollutant

10% removed in 5 hours $\Rightarrow$

when $t = 5$, $P = 90\% P_0$

$0.9 \cancel{P_0} = \cancel{P_0} e^{-k(5)}$

$e^{-5k} = 0.9$ / ln

$\ln e^{-5k} = \ln 0.9$

$-5k = \ln 0.9$

$k = \frac{-1}{5} \ln 0.9 \approx 0.02$

So $\boxed{P = P_0 e^{-0.02t}}$

(a) $t = 10$, $P = P_0 e^{-0.02(10)}$

$\frac{P}{P_0} = e^{-0.2} \approx 0.82 = 82\%$

After 10 hours, $P = 82\% P_0$

(b) $t = ?$ if $P = 50\% P_0$

$0.5 \cancel{P_0} = \cancel{P_0} e^{-0.02t}$

$e^{-0.02t} = 0.5$ / ln

$\ln e^{-0.02t} = \ln 0.5$

$-0.02t = \ln 0.5$

$t = \frac{\ln 0.5}{-0.02} \approx \boxed{34.6 \text{ hrs}}$

(15)

$A = P(1 + \frac{r}{n})^{nt}$

$P = 800$

$r = 0.04$

$n = 1$

$A = 2000$

$t = ?$

$2000 = 800(1 + \frac{0.04}{1})^t$

$(1 + 0.04)^t = \frac{20}{8}$

$(1.04)^t = \frac{5}{2}$ / ln

$\ln(1.04)^t = \ln 2.5$

$t \ln 1.04 = \ln 2.5$

$t = \frac{\ln 2.5}{\ln 1.04} \approx \boxed{23.4 \text{ years}}$

(16)

$f(t) = 11.65\left(1 - e^{-t/1.27}\right)$

t = time (seconds)

$f(t)$ = speed (m/sec)

(a) he crossed the finish line at $t = 9.86$ sec

$f(9.86) = 11.65\left(1 - e^{-\frac{9.86}{1.27}}\right)$

$\boxed{f(9.86) = 11.6451 \text{ m/sec}}$

his speed at the finish line

(b) $t = ?$ $f(t) = 10$ m/s

$10 = 11.65\left(1 - e^{-\frac{t}{1.27}}\right)$

$1 - e^{-\frac{t}{1.27}} = \frac{10}{11.65}$

$e^{-\frac{t}{1.27}} = 1 - \frac{10}{11.65}$

$e^{-\frac{t}{1.27}} = \frac{1.65}{11.65} = \frac{165}{1165}$ // ln

$\ln e^{-\frac{t}{1.27}} = \ln \frac{1650}{1165}$

$-\frac{t}{1.27} = \ln \frac{33}{233}$

$t = -1.27 \ln \frac{33}{233} \approx 2.4823$ seconds

He was running at 10 m/s after $\boxed{2.4823 \text{ seconds}}$.

(20)

$A = Pe^{rt}$

$P = 4$ \$

$A = 3(4) = 12$ \$

$r = 0.06$

$12 = 4e^{0.06t}$

$3 = e^{0.06t}$ // ln

$\ln 3 = \ln e^{0.06t}$

$0.06t = \ln 3$

$\boxed{t = \frac{\ln 3}{0.06} \approx 18.3 \text{ years}}$

6h

$9^x = 2e^{x^2}$ // ln

$\ln 9^x = \ln(2e^{x^2})$

$x \ln 9 = \ln 2 + \ln e^{x^2}$

$x \ln 9 = \ln 2 + x^2$

$x^2 - x\ln 9 + \ln 2 = 0$

quadratic equation with
$a=1,\ b=-\ln 9,\ c=\ln 2$

$$x = \frac{\ln 9 \pm \sqrt{(\ln 9)^2 - 4\ln 2}}{2}$$

$$x = \frac{\ln 9 \pm \sqrt{(\ln 9)^2 - \ln 16}}{2}$$

$x \simeq 1.81$

$x \simeq 0.38$

6m

$10^{2x} + 3(10^x) - 10 = 0$

let $10^x = t$

then $(10^x)^2 = t^2$

$10^{2x} = t^2$

The equation becomes:

$t^2 + 3t - 10 = 0$

$(t+5)(t-2) = 0$

$t = -5$ or $t = 2$

$t = -5$ or $t = 2$

$10^x = -5$ not possible

$10^x = 2$ // log

$\log 10^x = \log 2$

$x = \log 2$

6r

$\ln 5x - \ln(2x-1) = \ln 4$

Conditions $\begin{cases} 5x > 0 \\ \text{and} \\ 2x-1>0 \end{cases} \iff \begin{cases} x>0 \\ \text{and} \\ x > \frac{1}{2} \end{cases}$

therefore, $x > \frac{1}{2}$

$\ln \frac{5x}{2x-1} = \ln 4$

Natural log fct. is one-to-one

$\Rightarrow \frac{5x}{2x-1} = 4$

$5x = 4(2x-1)$

$5x = 8x - 4$

$3x = 4 \Rightarrow \boxed{x = \frac{4}{3}} > \frac{1}{2}$

12

let t = time (years)

M = quantity of material (kg)

t	M
0	20
5	$\frac{1}{2}(20)$
10	$(\frac{1}{2})^2(20)$
t	$(\frac{1}{2})^{t/5}20$

$\boxed{M = 20\left(\frac{1}{2}\right)^{\frac{t}{5}}}$

(a) $t = 10,\ M = \left(\frac{1}{2}\right)^2 20 = 5\,kg$

(b) $t = ?$ if $M = 0.1$

$0.1 = 20\left(\frac{1}{2}\right)^{\frac{t}{5}}$

$\left(\frac{1}{2}\right)^{\frac{t}{5}} = \frac{0.1}{20}$ // ln

$\ln\left(\frac{1}{2}\right)^{t/5} = \ln\frac{1}{200}$

$\frac{t}{5}\ln\frac{1}{2} = \ln\frac{1}{200}$

$t = \frac{5\ln\frac{1}{200}}{\ln 1/2} \approx 38.2$ years