# 3.2 Synthetic Division
# 3.3 Zeros of Polynomial Equations

In these sections we will study polynomials algebraically. Most of our work will be concerned with finding the solutions of polynomial equations of any degree – that is, equations of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0 \qquad (1)$$

**Definition** | A **root** or **solution** of equation (1) is a number k that when substituted for $x$ leads to a true statement. Thus, k is a root of equation (1) provided $f(k) = 0$.
We also refer to the number $c$ in this case as **a zero of the function** ***f***.

**Exercise #1** Checking for a zero or root.

a) Is -1 a zero of $P(x) = -x^3 + x^2 - x + 1$?

$x = -1$ is a zero if and only if $P(-1) = 0$

$P(-1) = -(-1)^3 + (-1)^2 - (-1) + 1 = 1+1+1+1 \neq 0$

Therefore, $-1$ is NOT a zero.

b) Is $x = \dfrac{1}{2}$ a root of the equation $2x^2 - 3x + 1 = 0$ ?

$x = \frac{1}{2}$ is a root if and only if $2\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 1 = 0$

$2 \cdot \frac{1}{4} - \frac{3}{2} + 1 = 0$

$\frac{1}{2} - \frac{3}{2} + 1 = 0$ TRUE

Therefore, $x = \frac{1}{2}$ is a root of the equation.

Note: If a root is repeated *n* times, we call it a **root of multiplicity *n***

**Exercise #2** a) State the multiplicity of each root of the equation: $x^2(x+1)^3(x-1) = 0$ =>

or $x^2 = 0 \Rightarrow x = 0$ root of multiplicity 2

$(x+1)^3 = 0 \Rightarrow x = -1$ root of multiplicity 3

or $x - 1 = 0 \Rightarrow x = 1$ root of multiplicity 1

(3.3 - #46) b) Find all zeros and their multiplicities: $f(x) = 5x^2\left(x + 1 - \sqrt{2}\right)(2x+5)$

$5x^2(x+1-\sqrt{2})(2x+5) = 0 \Rightarrow$

or $x^2 = 0 \Rightarrow x = 0$ root of multiplicity 2

or $x + 1 - \sqrt{2} = 0 \Rightarrow x = -1+\sqrt{2}$ root of multiplicity 1

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$ root of multiplicity 1

(3.3 - #48) c) Find all zeros and their multiplicities: $f(x) = (7x-2)^3(x^2+9)^2$

$(7x-2)^3(x^2+9)^2 = 0 \Rightarrow$

$(7x-2)^3 = 0 \Rightarrow 7x - 2 = 0 \Rightarrow x = \frac{2}{7}$ root of multiplicity 3

or $(x^2+9)^2 = 0 \Rightarrow x^2 + 9 = 0$

$x^2 = -9 \Rightarrow$ $x = 3i$ root of multiplicity 2

$x = \pm 3i$ $\quad x = -3i$ root of multiplicity 2

**Division of Polynomials**

The process of long division for polynomials follows the same four-step cycle used in ordinary long division of numbers: divide, multiply, subtract, bring down.
Notice that in setting up the division, we write both the dividend and divisor in decreasing powers of $x$.

**Example #1** Divide $5x^3 - 6x^2 - 28x - 2$ by $x + 2$.
(3.2 – Example 1)

$$\begin{array}{r l}
 & 5x^2 - 16x + 4 \\
x+2 \,\big)\!\! & 5x^3 - 6x^2 - 28x - 2 \\
 & -5x^3 - 10x^2 \\
 & \quad -16x^2 - 28x - 2 \\
 & \quad +16x^2 + 32x \\
 & \qquad 4x - 2 \\
 & \qquad -4x - 8 \\
 & \qquad \boxed{-10}\ \text{The Remainder}
\end{array}$$

$\frac{5x^3}{x} = 5x^2$

$\frac{-16x^2}{x} = -16x$

$\frac{4x}{x} = 4$

The result of the division can be written as: $\frac{5x^3-6x^2-28x-2}{x+2} = 5x^2 - 16x + 4 + \frac{-10}{x+2}$

or $\underbrace{5x^3-6x^2-28x-2}_{\substack{\text{DIVIDEND} \\ f(x)}} = \underbrace{(x+2)}_{\substack{\text{DIVISOR} \\ g(x)}}\underbrace{(5x^2-16x+4)}_{\substack{\text{QUOTIENT} \\ q(x)}} + \underbrace{(-10)}_{\substack{\text{REMAINDER} \\ r(x)}}$

Note 1) Second equation is valid for all real numbers $x$, whereas first equation carries implicit restrictions that $x$ my not equal -2. For this reason, we often prefer to write our results in the form of the second equation.

2) The degree of the remainder is less than the degree of the divisor. This is very similar to the situation with ordinary division of positive integers, where the remainder is always less than the divisor.

**The Division Algorithm**

(3.2)

Let $f(x)$ and $g(x)$ be polynomials with $g(x)$ of lower degree than $f(x)$ and assume that $g(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$ such that

$$\boxed{f(x) = g(x) \cdot q(x) + r(x)}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$. The polynomials $f(x)$ and $g(x)$ are called the **dividend** and **divisor**, respectively, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

When $r(x) = 0$, we have $f(x) = g(x) \cdot q(x)$ and we say that $g(x)$ and $q(x)$ are **factors** of $f(x)$.

**Exercise #4** Using long division to find a quotient and a remainder.

Divide $x^3+2x^2-4$ by $x-3$.

$$\begin{array}{r|l} & x^2+5x+15 \\ x-3 & x^3+2x^2+0x-4 \\ & -x^3+3x^2 \\ \hline & \quad 5x^2+0x-4 \\ & \quad -5x^2+15x \\ \hline & \qquad 15x-4 \\ & \qquad -15x+45 \\ \hline & \qquad \boxed{41}\ R \end{array}$$

$\frac{x^3}{x}=x^2$

$\frac{5x^2}{x}=5x$

$\frac{15x}{x}=15$

$x^3+2x^2-4=(x-3)(x^2+5x+15)+41$

$x^2+5x+15=$ quotient ; $41=$ remainder

**Synthetic Division**

- Synthetic division is a quick method of dividing polynomials.
- It can be used **when the divisor is of the form $x-k$.**
- In the synthetic division we write down only the essential parts of the long division table (the coefficients).

**Exercise #5** Use synthetic division to perform the following divisions:
(3.2 - #2, 11)

a) $\dfrac{x^3+4x^2-5x+44}{x+6}$

$x+6=x-(-6)$

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & 44 \\ \hline -6 & \underbrace{1 \quad -2 \quad 7}_{\text{coefficients of quotient}} & & & \boxed{2}\ R \end{array}$$

$\dfrac{x^3+4x^2-5x+44}{x+6}=x^2-2x+7+\dfrac{2}{x+6}$

OR $x^3+4x^2-5x+44=(x+6)(x^2-2x+7)+2$ (*)

If $f(x)=x^3+4x^2-5x+44$, evaluate $f(-6)$. What do you observe?

From (a), $f(x)=(x+6)(x^2-2x+7)+2$

$f(-6)=0+2=2$

We see that $f(-6)=2=$ the remainder when dividing $f(x)$ by $x+6$

b) $\dfrac{\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1}{x-\frac{1}{3}}$ $= \frac{1}{3}x^2 - \frac{1}{9}x + \frac{1}{x-\frac{1}{3}}$

$$\begin{array}{c|cccc} & \frac{1}{3} & -\frac{2}{9} & \frac{1}{27} & 1 \\ \hline \frac{1}{3} & \underbrace{\frac{1}{3} \quad -\frac{1}{9} \quad 0}_{\text{coefficients of quotient}} & & & \boxed{1}\,R \end{array}$$

If $f(x)=\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1$, evaluate $f\left(\frac{1}{3}\right)$. What do you observe?

From (b), $f(x) = (x-\frac{1}{3})(\frac{1}{3}x^2-\frac{1}{9}x)+1$

$f(\frac{1}{3}) = 0+1 = 1$ = the remainder when $f(x)$ is divided by $x-\frac{1}{3}$.

**The Remainder Theorem** (3.2)

| When we divide a polynomial $f(x)$ by $x-k$, the remainder is $f(k)$. |
|---|

Proof

From the division algorithm $\Rightarrow f(x) = (x-k)\,q(x) + r(x)$
where $r(x)=0$ or degree $r(x)$ < degree $(x-k)$
degree $r(x) < 1 \Rightarrow r(x) = \text{constant} = r$

So, $f(x) = (x-k)\,q(x) + r$
$f(k) = 0 \cdot q(k) + r \Rightarrow \boxed{f(k) = r}$

**Exercise #6** (3.2 - #27, 35) Using the remainder theorem to evaluate a function and check for a factor.

a) Let $f(x)=x^2+5x+6$.

i) Evaluate $f(-2)$.

ii) Is $x+2$ a factor of $f(x)=x^2+5x+6$?

(i) $f(-2)$ = remainder when dividing $f(x)$ by $x-(-2) = x+2$

$$\begin{array}{c|ccc} & 1 & 5 & 6 \\ \hline -2 & 1 & 3 & \boxed{0}\,R \end{array} \Rightarrow \boxed{f(-2)=0}$$

(ii) yes, $x+2$ is a factor of $f(x)$ $\qquad x+2 \mid f(x)$

$f(x) = (x+2)(x+3)$

b) Let $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6$.

i) Evaluate $f\left(\frac{1}{2}\right)$.

ii) Is $x - \frac{1}{2}$ a factor of $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6$?

(i)
$$\begin{array}{c|ccccc} & 6 & 1 & -8 & 5 & 6 \\ \hline \frac{1}{2} & 6 & 4 & -6 & 2 & \boxed{7} \end{array} \Rightarrow f\left(\tfrac{1}{2}\right) = 7$$
(Remainder th)

(ii) $x - \frac{1}{2} \nmid f(x)$

($x - \frac{1}{2}$ doesn't divide $f(x)$
$x - \frac{1}{2}$ is not a factor of $f(x)$)

**The Factor Theorem** (3.3)

| The polynomial $x - k$ is a factor of the polynomial $f(x)$ if and only if $f(k) = 0$. |
|---|

**Exercise #7** Let $f(x) = 2x^3 - 4x^2 + 2x - 1$.

a) What is the remainder when dividing the given polynomial by $x - 2$? In how many ways can you find the remainder? Which method is the easiest one?

The remainder could be found using long division or synthetic division or The Remainder theorem

$$\begin{array}{c|cccc} & 2 & -4 & 2 & -1 \\ \hline 2 & 2 & 0 & 2 & \boxed{3} \end{array}$$

$R = 3$

b) Is $x - 2$ a factor of $f(x)$?

No, $x - 2 \nmid f(x)$

c) Is $x - 1$ a factor of $f(x)$? Using the Factor theorem,

$x - 1 \mid f(x)$ iff $f(1) = 0$

$f(1) = 2 - 4 + 2 - 1 \neq 0$

Therefore $x - 1 \nmid f(x)$

**Exercise #8** Factoring a polynomial given a zero.

(3.3 - #19) a) Let $f(x) = 6x^3 + 13x^2 - 14x + 3$. Show that -3 is a zero and use this fact to factor $f(x)$ completely.

$x = -3$ is a zero iff $f(-3) = 0$

| | | | | |
|---|---|---|---|---|
| | 6 | 13 | -14 | 3 |
| -3 | 6 | -5 | 1 | $\boxed{0}$ R |

$\Rightarrow x+3 \,/\, f(x)$

$f(x) = (x+3)(6x^2 - 5x + 1)$

$f(x) = (x+3)(3x-1)(2x-1)$

(3.3 - #28) b) $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5$. Knowing that -1 is a root of multiplicity 3, factor $f(x)$ into linear factors.

$x = -1$ root of multiplicity 3 $\Rightarrow (x+1)^3 / f(x)$

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 1 | -9 | -13 | -5 | |
| -1 | 2 | -1 | -8 | -5 | 0 | $\Rightarrow f(x) = (x+1)(2x^3 - x - 8x - 5)$ |
| -1 | 2 | -3 | -5 | 0 | | $\Rightarrow f(x) = (x+1)^2(2x^2 - 3x - 5)$ |
| -1 | 2 | -5 | 0 | | | $\Rightarrow f(x) = (x+1)^3(2x-5)$ |

**The Conjugate Zeros Theorem**
(3.3)

> If $f(x)$ is a polynomial function with real coefficients and if $a+bi$ is a zero of $f(x)$, then its conjugate $a-bi$ is also a zero of $f(x)$.

**Exercise #9** For each polynomial , one zero is given. Find all the others.
(3.3 - #31, 32)

a) $f(x) = x^3 - 7x^2 + 17x - 15$ ; $2-i$

$x = 2-i \Rightarrow x-(2-i) \,/\, f(x)$
then
$x = 2+i \Rightarrow x-(2+i) \,/\, f(x)$

$\Rightarrow (x-2+i)(x-2-i) \,/\, f(x)$

$((x-2)^2 - i^2) \,/\, f(x)$

$x^2 - 4x + 5 \,/\, f(x)$

$$
\begin{array}{r}
x - 3 \\
x^2-4x+5 \enclose{longdiv}{x^3 - 7x^2 + 17x - 15} \\
-x^3 + 4x^2 - 5x \\
\hline
/ \quad -3x^2 + 12x - 15 \\
3x^2 - 12x + 15 \\
\hline
/
\end{array}
$$

so $f(x) = (x-3)(x^2 - 4x + 5)$

The zeros are $\boxed{\begin{array}{l} x = 2-i \\ x = 2+i \\ x = 3 \end{array}}$

b) $f(x)=4x^3+6x^2-2x-1;\ \frac{1}{2}$

$x=\frac{1}{2}$ zero $\Leftrightarrow f(\frac{1}{2})=0 \Leftrightarrow x-\frac{1}{2} \mid f(x)$

| | 4 | 6 | -2 | -1 |
|---|---|---|---|---|
| $\frac{1}{2}$ | 4 | 8 | 2 | 0 |

$f(x)=(x-\frac{1}{2})(4x^2+8x+2)$

$f(x)=2(x-\frac{1}{2})(2x^2+4x+1)$

$\boxed{x=\frac{1}{2}}$ OR $2x^2+4x+1=0$

$x=\frac{-4\pm\sqrt{16-8}}{4}=\frac{-4\pm 2\sqrt{2}}{4}=\frac{-2\pm\sqrt{2}}{2}$

The zeros are $\boxed{\{\frac{1}{2}, \frac{-2\pm\sqrt{2}}{2}\}}$

**The Fundamental Theorem of Algebra**
(3.3)

> Every polynomial equation of the form
> $$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0=0 \quad (n\ge 1, a_n\ne 0)$$
> has at least one root within the complex number system. (This root may be a real number).

**The Linear Factors Theorem**

> Every polynomial of degree $n$ can be expressed as a product of $n$ linear factors.
> $$f(x)=a_n(x-x_1)(x-x_2)\ldots(x-x_n),$$
> where $a_n$ is the leading coefficient and $x_i$ are the roots of the polynomial.

**Theorem**

> Every polynomial of degree $n\ge 1$ has exactly n roots, where a root of multiplicity $k$ is counted $k$ times.

**Exercise #10** Write each polynomial as a product of linear factors.

a) $f(x)=3x^2-5x-2$

Method I Factoring directly

$\boxed{f(x)=(3x+1)(x-2)}$

Method II - Using the Linear Factors th.

Find all zeros: $3x^2-5x-2=0$

$x=\frac{5\pm\sqrt{25+24}}{6}=\frac{5\pm 7}{6}$

$\begin{cases} x=2 \\ x=-\frac{1}{3}\end{cases}$

$\boxed{f(x)=3(x-2)(x+\frac{1}{3})}$

b) $f(x) = x^2 - 5$

Find all zeros:

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5}$

Then, $f(x) = 1(x - \sqrt{5})(x + \sqrt{5})$

$\boxed{f(x) = (x - \sqrt{5})(x + \sqrt{5})}$

c) $f(x) = x^2 - 4x + 5$

Find all zeros:

$x^2 - 4x + 5 = 0$

$x = \dfrac{4 \pm \sqrt{16 - 20}}{2}$

$x = \dfrac{4 \pm 2i}{2}$ $\begin{cases} x = 2 + i \\ x = 2 - i \end{cases}$

Then, $f(x) = 1(x - (2+i))(x - (2-i))$

$\boxed{f(x) = (x - 2 - i)(x - 2 + i)}$.

**Exercise #11** Finding polynomial equations satisfying given conditions.

In each case, find a polynomial equation $f(x) = 0$ satisfying the given conditions. If there is no such equation, say so.

(3.3 - #49) a) Find a polynomial function of degree 3 having the numbers -3, 1, and 4 as roots and satisfying $f(2) = 30$.

$x = -3$ zero $\xRightarrow{\text{Factor theorem}}$ $x + 3 \mid f(x)$

$x = 1$ zero $\Rightarrow$ $x - 1 \mid f(x)$

$x = 4$ zero $\Rightarrow$ $x - 4 \mid f(x)$

degree $f(x) = 3 \Rightarrow x+3$, $x-1$, and $x-4$ are the only factors

So, $f(x) = a(x+3)(x-1)(x-4)$

$\Rightarrow$ But, $f(2) = 30$

$30 = a(5)(1)(-2)$

$30 = -10a \Rightarrow a = -3$

Therefore,

$\boxed{f(x) = -3(x+3)(x-1)(x-4)}$

b) A factor of $f(x)$ is $x - 3$, and -4 is a root of multiplicity 2.

$x - 3 \mid f(x)$

$x = -4$ zero of m.=2 $\Rightarrow$ $(x+4)^2 \mid f(x)$ $\Rightarrow$

The simplest polynomial equation is

$\boxed{f(x) = (x-3)(x+4)^2 = 0}$

(3.3 - #53) c) Find a polynomial function of degree 3 having the number -3 as a zero of multiplicity 3 and satisfying the condition $f(3) = 36$.

$x = -3$ zero of $m = 3 \Rightarrow (x+3)^3 \mid f(x)$

degree $f(x) = 3 \Rightarrow (x+3)$ is the only factor

so $f(x) = a(x+3)^3$

But $f(3) = 36$

$36 = a(6)^3 \Rightarrow a = \frac{1}{6}$

Therefore, $\boxed{f(x) = \frac{1}{6}(x+3)^3}$

**Exercise #12** (3.3 - #57, 68) Find a polynomial function of least degree having only real coefficients with zeros as given. What is the degree of each polynomial?

a) 2 and $1+i$.

$x = 2$ zero $\Rightarrow x - 2 \mid f(x)$

$x = 1+i$ zero $\Rightarrow x - (1+i) \mid f(x)$

From the Conjugate zeros theorem $\Rightarrow$

$x = 1-i$ zero $\Rightarrow x - (1-i) \mid f(x)$

degree $f(x) = 3$

$f(x) = (x-2)(x-(1+i))(x-(1-i))$

$f(x) = (x-2)(x-1-i)(x-1+i)$

$f(x) = (x-2)((x-1)^2 - i^2)$

$f(x) = (x-2)(x^2 - 2x + 1 - (-1))$

$\boxed{f(x) = (x-2)(x^2 - 2x + 2)}$

b) $5+i$ and $4-i$.

given $x = 5+i$ zero $\Rightarrow x - (5+i) \mid f(x)$

From the Conjugate zeros theorem $\Rightarrow$

$x = 5-i$ zero $\Rightarrow x - (5-i) \mid f(x)$

given $x = 4-i$ zero $\Rightarrow x - (4-i) \mid f(x)$

From Conjugate zeros theorem $\Rightarrow$

$x = 4+i$ zero $\Rightarrow x - (4+i) \mid f(x)$

degree $f(x) = 4$

$f(x) = (x-5-i)(x-5+i)(x-4+i)(x-4-i)$

$f(x) = ((x-5)^2 - i^2)((x-4)^2 - i^2)$

$f(x) = (x^2 - 10x + 25 - (-1))(x^2 - 8x + 16 - (-1))$

$\boxed{f(x) = (x^2 - 10x + 26)(x^2 - 8x + 17)}$

**The Number and Location of Real Zeros**

**Descartes' Rule of Signs**

In some cases, the following rule – discovered by the French philosopher and mathematician Rene Descartes around 1637 – is helpful in eliminating candidates from lengthy lists of possible rational roots.

To describe this rule, we need the concept of **variation in sign**. If $f(x)$ is a polynomial with real coefficient, written with descending powers of $x$ (and omitting powers with coefficient 0), then a variation in sign is a change from positive to negative or negative to positive in successive terms of the polynomial (adjacent coefficients have opposite signs).

**Example #5** How many variations in sign occur in the following polynomial?

$$f(x) = \underbrace{5x^7 - 3x^5}_{1} \underbrace{- x^4 + 2x^2}_{2} \underbrace{+ x - 3}_{3}$$

There are 3 variations in sign.

**Descartes'Rule of Signs**
(3.3)

> Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.
>
> a) The number of positive real zeros of $f(x)$ is either equal to the number of variations in sign in $f(x)$ or is less than that by an even whole number.
>
> b) The number of negative real zeros of $f(x)$ is either equal to the number of variations in sign in $f(-x)$ or is less than that by an even whole number.

**Exercise #13** Use Descartes' rule of signs to determine the possible number of positive real zeros and
(3.3 - #73, #77) negative real zeros for each function.

a) $f(x) = \underbrace{2x^3 - 4x^2}_{1} \underbrace{+ 2x}_{2} + 7$

# positive zeros

There are 2 variations in sign in $f(x)$

2 positive real zeros
OR
0 positive real zeros

# negative zeros

$f(-x) = -2x^3 - 4x^2 \underbrace{-2x + 7}$

There is 1 variation in sign in $f(-x)$ — 1 negative real zeros

b) $f(x) = x^5 \underbrace{+3x^4 - x^3}_{1} \underbrace{+2x}_{2} + 3$

2 variations in sign in $f(x)$ ⟹ $\begin{cases} 2 \text{ positive real zeros} \\ \text{OR} \\ 0 \text{ positive real zeros} \end{cases}$

$f(-x) = \underbrace{-x^5 + 3x^4}_{1} \underbrace{+ x^3 - 2x}_{2} \underbrace{+3}_{3}$

3 variations in sign in $f(-x)$ ⟹ $\begin{cases} 3 \text{ negative real zeros} \\ \text{OR} \\ 1 \text{ negative real zero} \end{cases}$

**<u>Finding all the rational zeros of a polynomial</u>**

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. We now study a method for finding all the <u>rational</u> zeros of a polynomial.

**<u>Example #3</u>** Consider the polynomial

$f(x) = (x-2)(x-3)(x+4)$ Factored form

$= x^3 - x^2 - 14x + 24$ Expanded form.

What are the zeros of $f(x)$? 2, 3, −4

What relationship exists between the zeros and the constant term of the polynomial?

2, 3, −4 are factors of 24

The next theorem generalizes this observation.

**<u>The Rational Zeros Theorem</u>**
(3.3)

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ ($a_0 \neq 0, a_n \neq 0$) has <u>integer coefficients</u>, then every **rational zero** of $f(x)$ is of the form $\frac{p}{q}$ where $p$ is a factor of the constant coefficient $a_0$

$q$ is a factor of the leading coefficient $a_n$.

<u>Note</u>: The Rational Zeros Theorem gives only POSSIBLE rational zeros. It does not tell us whether these rational numbers are actual zeros.

**<u>Exercise #14</u>** <u>Using the Rational Zeros Theorem</u>
(3.3 – Example 3)

Do each of the following for the polynomial function defined by

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$$

a) List all possible rational zeros.

$\frac{p}{q} = \frac{\text{factor of } 2}{\text{factor of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6}$

$\boxed{\frac{p}{q} \in \left\{ \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3} \right\}}$

b) Find all rational zeros and factor $f(x)$ into linear factors.

We see that $f(1) = 0 \Rightarrow x - 1 \mid f(x)$

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ \hline & 6 & 13 & 1 & -2 & 0 \end{array}$$

$f(x) = (x-1)(6x^3 + 13x^2 + x - 2)$

Factor $6x^3 + 13x^2 + x - 2$

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

$f(x) = (x-1)(x+2)(6x^2 + x - 1)$

$\boxed{f(x) = (x-1)(x+2)(3x-1)(2x+1)}$

**Finding the Rational Zeros of a Polynomial**

1. List all possible rational zeros using the Rational Zeros Theorem.
2. Use synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. when the remainder is 0, note the quotient you have obtained.
3. Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is a quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

**Exercise #15** For each polynomial function
(3.3 - #37, 40)

i) List the maximum number of real zeros;
ii) List the number of positive real zeros and negative real zeros; (Descartes' Rule)
iii) list all possible rational zeros; (Rational zeros theorem)
iv) find all rational zeros; (Factor theorem & Synthetic Division)
v) factor $f(x)$.

a) $f(x)=x^3+6x^2-x-30$.

i) max # real zeros = 3
ii) # positive real zeros = 1 (there is one variation in sign in $f(x)$)
\# negative real zeros = 2 or 0 (there are two variations in sign in $f(-x)$)

$$f(-x) = \underbrace{-x^3+6x^2}_{1}\underbrace{+x-30}_{2}$$

iii) possible rational zeros $\frac{p}{q} = \frac{\text{factor of } 30}{\text{factor of } 1}$

$$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30\}$$

iv) We note that $f(1) \neq 0$ so $x=1$ not a zero

Try $x=2$

$$\begin{array}{c|cccc} & 1 & 6 & -1 & -30 \\ \hline 2 & 1 & 8 & 15 & 0 \end{array}$$

$f(x) = (x-2)(x^2+8x+15)$

$f(x) = (x-2)(x+5)(x+3)$

The zeros are $\begin{cases} x=2 \\ x=-5 \\ x=-3 \end{cases}$

v) $f(x) = (x-2)(x+5)(x+3$

b) $f(x)=15x^3+61x^2+2x-8$

i) max # real zeros = 3

ii) # positive real zeros = 1 (there is one variation in sign in $f(x)$)

# negative real zeros = 2 or 0 (there are two variations in sign in $f(-x)$)

$f(-x)=-15x^3+61x^2-2x-8$

iii) Possible rational zeros

$$\frac{p}{q}=\frac{\text{factor of } 8}{\text{factor of } 15}=\frac{\pm1,\pm2,\pm4,\pm8}{\pm1,\pm3,\pm5,\pm15}$$

$$\frac{p}{q}\in\left\{\pm1,\pm2,\pm4,\pm8,\pm\frac{1}{3},\pm\frac{1}{5},\pm\frac{1}{15},\pm\frac{2}{3},\pm\frac{2}{5},\pm\frac{2}{15},\pm\frac{4}{3},\pm\frac{4}{5},\pm\frac{4}{15},\pm\frac{8}{3},\pm\frac{8}{5},\pm\frac{8}{15}\right\}$$

iv) We see that $f(1)\neq 0$ (so $x=1$ is not a zero)
$f(-1)\neq0$ (so $x=-1$ is not a zero)

| | | | | |
|---|---|---|---|---|
| $-4$ | 15 | 61 | 2 | $-8$ |
| | 15 | 1 | $-2$ | 0 |

$f(x)=(x+4)(15x^2+x-2)$

$f(x)=(x+4)(3x-1)(5x+2)$ (v)

All zeros are $x=-4$, $x=\frac{1}{3}$, $x=-\frac{2}{5}$.

**Exercise #16** a) Find all the complex zeros of $f(x)=x^4-6x^3+22x^2-30x+13$.

b) Find all the solutions of $x^4-5x^3-5x^2+23x+10=0$.

(a) $f(x)=x^4-6x^3+22x^2-30x+13$

possible rational zeros $\frac{p}{q}=\frac{\text{factor } 13}{\text{factor } 1}\in\{\pm1,\pm13\}$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | $-6$ | 22 | $-30$ | 13 |
| | 1 | $-5$ | 17 | $-13$ | 0 |

$f(x)=(x-1)(\underline{x^3-5x^2+17x-13})$

factor it! Possible rational zeros $\pm1,\pm13$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | $-5$ | 17 | $-13$ |
| | 1 | $-4$ | 13 | 0 |

$f(x)=(x-1)(x-1)(x^2-4x+13)$

$x^2-4x+13=0$

$$x=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm6i}{2}=2\pm3i$$

The zeros are:

$x=1$ with multiplicity 2

$x=2+3i$, $x=2-3i$ multiplicity 1