5.6 Systems of Inequalities and Linear Programming

Systems of Inequalities

In class work:

Graph the solution set of each system of inequalities:

a) $\begin{cases} x+y\le 4 \\ x-2y>6 \end{cases}$ b) $\begin{cases} y\ge 3^x \\ y>2 \end{cases}$ c) $\begin{cases} y\le x^x \\ x^2+y^2<1 \end{cases}$ d) $\begin{cases} e^{-x}-y<1 \\ x-2y\ge 4 \end{cases}$

e) $\begin{cases} x<4 \\ x\ge 0 \\ y\ge 0 \\ x+2y>2 \end{cases}$ f) $\begin{cases} \ln x-y\ge 1 \\ x^2-2x-y\le 1 \end{cases}$ g) $\begin{cases} y\ge (x-2)^2+3 \\ y<-(x-1)^2+6 \end{cases}$

h) $\begin{cases} y\le x^3-x \\ y>-3 \end{cases}$ i) $\begin{cases} y\le \log x \\ y>|x-2| \end{cases}$ j) $\begin{cases} y\le \left(\frac{1}{2}\right)^x \\ y\ge 4 \end{cases}$

Linear Programming

An important application of mathematics to business and social science is called linear programming. Linear programming was first developed during and shortly after World War II. It has changed the way businesses and governments make decisions from guess-work to using an algorithm based on available data and guaranteed to produce an optimal decision.

Mixture Problems

In a mixture problem, limited resources are combined into products so that the profit from selling them is maximum. How should the available resources be shared among the possible products so that the profit is maximized?

Linear programming is a management science technique that helps a business allocate the resources on hand to make a particular mix of products that will maximize profit.
Linear programming is a tool for maximizing or minimizing a quantity, typically profit or cost subject to constraints.

What does it mean to find a solution to a linear programming mixture problem?

A solution to a mixture problem is a production policy that tells us how many units of each product to make.

The optimal production policy must be possible and must give the maximum profit.

Example 1 - Mixture problem having one resource

Suppose a toy manufacturer has 60 containers of plastic and wants to make and sell skateboards. The "recipe" for one skateboard requires five containers of plastic, plus paint and decals, which for simplicity we assume are available in essentially unlimited quantities. The profit on one skateboard is $1.00, and in order to keep things simple, we assume that there will be customers for every skateboard produced. So the manufacturer must decide how many skateboards to make.

Solution

Let x be the number of skateboards.

We see that the manufacturer can make ____________________ skateboards.

The profit earned is _________________ dollars.

Write an inequality to represent the number of skateboards produced: ____________________________________

These values are the feasible set, the particular values of our variable x that are feasible, or possible, given the available resources.

Graph the feasible set:

Definition The **feasible set** (feasible region) is the set of all possible solutions to a linear programming problem.

There are several features to note about our feasible region and the points within it that give the maximum profit:

1. There are no negative values of x in the feasible region.
2. Any point within the feasible region represents a possible production policy – that is, it gives the number of skateboards (product) that is possible to produce with the limited supply of containers of plastic (resources).
3. The point $x = 0$ of the feasible region represents the manufacturer making no skateboards at all, having no products to sell.
4. The point where the profit is greatest, $x = 12$, happens to be an endpoint, or "corner", of the feasible region.

The Corner Point Principle (The Fundamental Theorem of Linear Programming)

In a linear programming problem, the maximum value for the profit formula corresponds to a corner point of the feasible region.

Example 2 – Two products and one resource

A clothing company has 60 yards of cloth available to make shirts and vests. Each short requires 3 yards of material and provides a profit of $5.00. each vest requires 2 yards of material and provides a profit of $2.00. What is the maximum profit?

Solution

Let x be the number of shirts produced. Let y be the number of vests produced.

Resources – yards of cloth : 60

	Material	Profit
Shirts (x units)		
Vests (y units)		

Solving a Linear Programming Problem

Step 1 Write the objective function and all necessary constraints.

Step 2 Graph the region of feasible solutions.

Step 3 Identify all vertices or corner points.

Step 4 Find the value of the objective function at each vertex.

Step 5 The solution is given by the vertex producing the optimal value of the objective function.

Problem 1 An office manager wants to buy some filing cabinets. He knows that cabinet A costs $10 each, requires 6 ft^2 of floor space, and holds 8 ft^3 of files. Cabinet B costs $20 each, requires 8 ft^2 of floor space, and holds 12 ft^3. He can spend no more than $140 due to budget limitations, and his office has room for no more than 72 ft^2 of cabinets. He wants to maximize storage capacity within the limits imposed by funds and space. How many of each type of cabinet should he buy?

(Answer: 8 type A and 3 type B)

Problem 2 The manufacturing process requires that oil refineries manufacture at least 2 gal of gasoline for each gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal per day must be produced. The demand for gasoline is no more than 6.4 million gal per day. If the price of gasoline is $1.90 per gal and the price of fuel oil is $1.50 per gal, how much of each should be produced to maximize revenue?

(Answer: 6.4 mil gallons of gasoline and 3.2 million gallons of fuel)

Problem 3 A company makes two products: MP3 players and DVD players. Each MP3 gives a profit of $30, while each DVD produces $70 profit. The company must manufacture at least 10 MP3s per day to satisfy one of its customers, but no more than 50 because of production problems. The number of DVD players produced cannot exceed 60 per day, and the number of MP3s cannot exceed the number of DVD players. How many of each should the company manufacture to obtain maximum profit?

(Answer: 50 MP3s and 60 DVD2)

Problem 4 Robin takes vitamin pills each day. She wants at least 16 units of Vitamin A, at least 5 units of Vitamin B1, and at least 20 units of Vitamin C. She can choose between red pills, costing 10 cents each, that contain 8 units of A, 1 of B1, and 2 of C; and blue pills, costing 20 cents each, that contain 2 units of A, 1 of B1, and 7 of C. How many of each should she buy to minimize her cost and yet fulfill her daily requirements?

(Answer: 3 red pills and 2 blue pills, for a total of 70 cents per day)