Section 4.2
Exponential Functions

Review

Complete the following:

$a^{\frac{m}{n}} =$

1) Write using radical notation:

a) $10^{\frac{4}{5}}$ b) $x^{\frac{3}{7}}$

2) Use exponent notation:

a) $\sqrt[7]{(1+n)^4}$ b) $\sqrt[3]{x^5}$

3) Simplify :

a) $16^{\frac{1}{2}}$ b) $(-32)^{\frac{1}{5}}$ c) $64^{-\frac{1}{2}}$

4) If $f(x) = 3^x$, find each of the following:

a) $f(2)$ b) $f(-3)$

5) Solve the following equations:

a) $3^x = 27$ b) $2^{3y+1} = \sqrt{2}$ c) $\left(\frac{1}{2}\right)^k = 4$

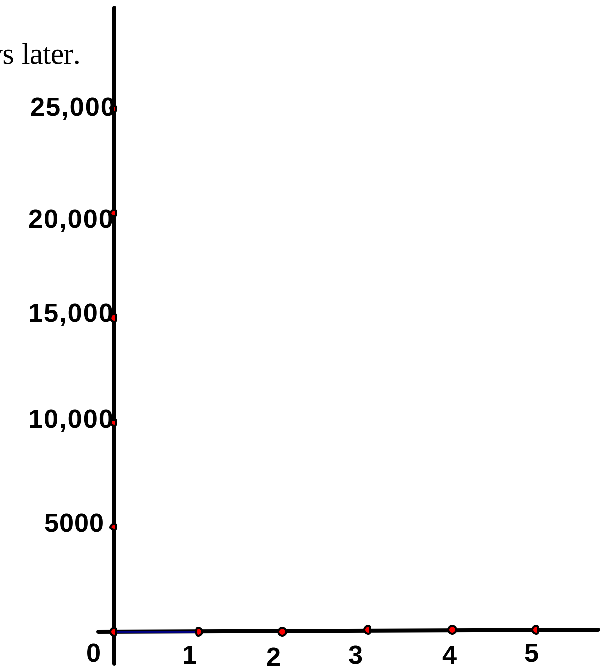
d) $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ e) $\left(\sqrt[3]{5}\right)^{-x} = \left(\frac{1}{5}\right)^{x+2}$

Exponential Growth and Decay

Example 1 In a laboratory experiment, researchers establish a colony of 100 bacteria and monitor its growth. The experiments discover that the colony triples in population every day.

1. Fill in Table 1. showing the population, $P(t)$, of bacteria t days later.

t	P(t)
0	
1	
2	
3	
4	
5	



2. Plot the data points from Table 1. and connect them with a smooth curve.

3. Write a function that gives the population of the colony at any time t in days. (Express the values you calculated in part (1) using powers of 3. Do you see a connection between the value of t and the exponent on 3?)

4. Evaluate your function to find the number of bacteria present after 8 days. How many bacteria are present after 36 hours?

Growth Factors

The function in Example 1 describes **exponential growth.** During each time interval of a fixed length the population is multiplied by a certain constant amount. The bacteria population grows by a factor of 3 every day. For this reason we say that 3 is the **growth factor** for the function. Functions that describe exponential growth can be expressed in the standard form

$$\boxed{P(t) = P_0 a^t}$$

where $P_0 = P(0)$ is the initial value of the function and a is the growth factor.

For the bacteria population we have $P(t) = 100 \cdot 3^t$ so $P_0 = 100$ and $a = 3$

Exercise #1 A lab technician compares the growth of two species of bacteria. She starts two colonies of 50 bacteria each. Species A doubles in population every 2 days, and species B triples every 3 days. Find the growth factor for each species. Which species grows faster?

Example 2 **Percent Increase**

Exponential growth occurs in other circumstances, too. For example, of the interest on a savings account is compounded annually, the amount of money in the account grows exponentially.

Consider a principal of $100 invested at 5% interest compounded annually.

What is the amount in the account at the end of 1 year?
Write the formula for the amount in factored form.

t	A(t)
0	
1	
2	
3	

The amount, $105, becomes the new principal for the second year.
To find the amount at the end of the second year, we apply the formula again, with $P = 105$.
Observe that to find the amount at the end of each year we multiply the principal by a factor of $1 + r = 1.05$.

A formula for the amount after *t* years is $A(t) = 100(1.05)^t$.

In general, for an initial investment of *P* dollars at an interest rate , *r*, compounded annually, the amount accumulated after *t* years is

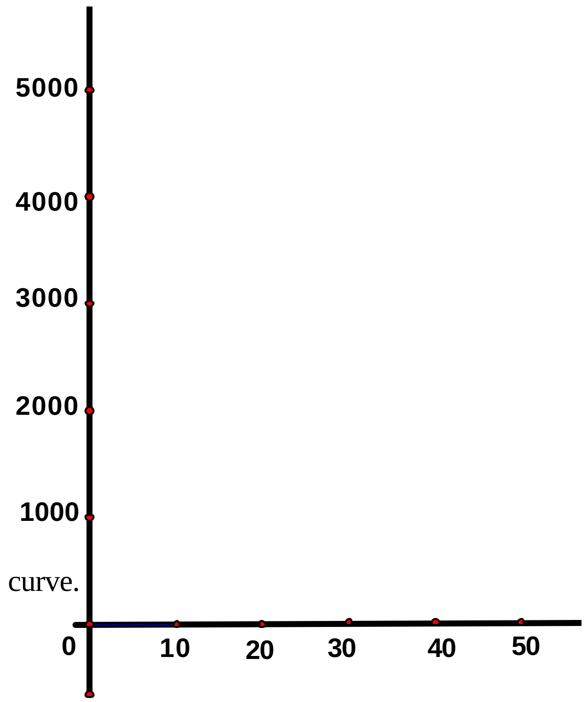
$$\boxed{A(t) = P(1+r)^t}$$

This function describes exponential growth with an initial value of P and a growth factor of $a = 1 + r$.
The interest rate, *r,* which indicates the *percent increase* in the account each year, corresponds to a *growth factor* of $a = 1 + r$.

Example 3 A small coal-mining town has been losing population since 1940, when 5000 people lived there. At each census thereafter (taken at 10-year intervals) the population has declined to approximately 0.90 of its earlier figure.

1. Fill in Table 2. showing the population, $P(t)$, of the town t years after 1940.

t	P(t)
0	
10	
20	
30	
40	
50	



2. Plot the data points from Table 2. and connect them with a smooth curve.

3. Write a function that gives the population of the town at any time t in years after 1940. (Express the values you calculated in part (1) using powers of 0.90. Do you see a connection between the value of t and the exponent on 0.90?)

4. Evaluate your function to find the population of the town in 1995. What was the population in 2000?

Exponential Growth and Decay Functions

The function

$$P(t) = P_0 a^t$$

models exponential growth and decay.

$P_0 = P(0)$ is the **initial value** of P;
a is the **growth** or **decay factor**.

1. If $a > 1$, $P(t)$ is increasing, and $a = 1 + r$, where r represents percent increase.
2. If $0 < a < 1$, $P(t)$ is decreasing, and $a = 1 - r$, where r represents percent decrease.

The Exponential Function: $\boxed{f(x) = ab^x \text{, where } b > 0 \text{ and } b \neq 1, a \neq 0 \text{.}}$

a = the coefficient, b = base.

Note: If $b = 1$, then $b^x = 1^x = 1$ for any x – trivial.

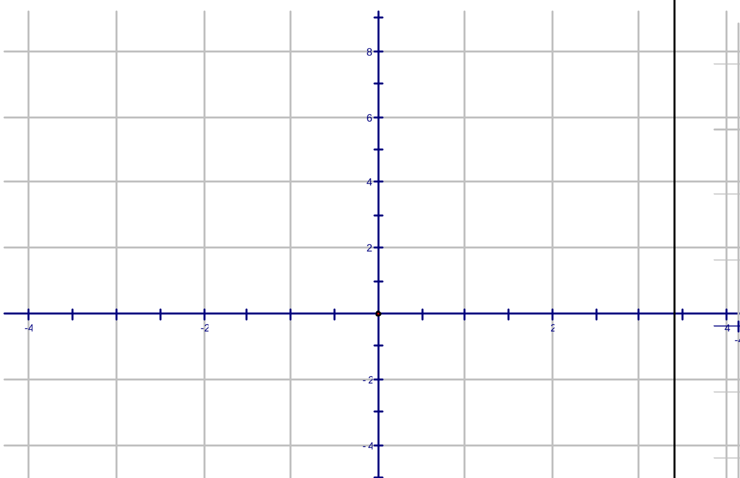
If $b = 0$, then $b^x = 0^x$ which is undefined when $x = 0$

If $b < 0$, let $f(x) = (-2)^x$. If $x = \frac{1}{2}$, $f(x) = (-2)^{\frac{1}{2}} = \sqrt{-2} \notin \mathbb{R}$

Graphs of Exponential Functions

$$f(x) = 2^x$$

x	$-\infty$	∞
y		



Domain: ____________________

Range: ____________________

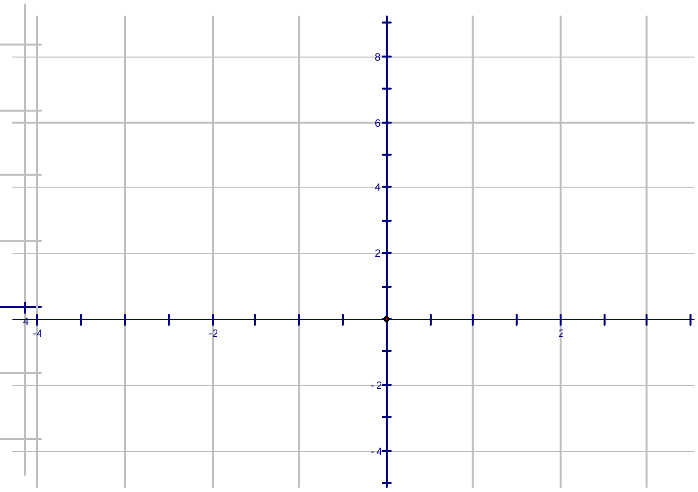
Horizontal Asymptote: ____________________

If $b > 1$, the function is **increasing**.

The function is one-to-one.

$$g(x) = \left(\frac{1}{2}\right)^x$$

x	$-\infty$	∞
y		



Domain: ____________________

Range: ____________________

Horizontal Asymptote: ____________________

If $0 < b < 1$, the function is **decreasing.**

The function is one-to-one.

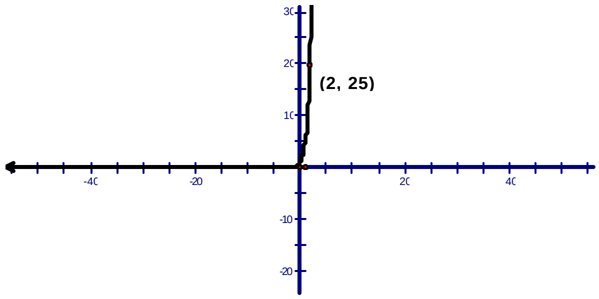
Question: Which function grows more rapidly: $y = 3^x$ or $y = 4^x$?

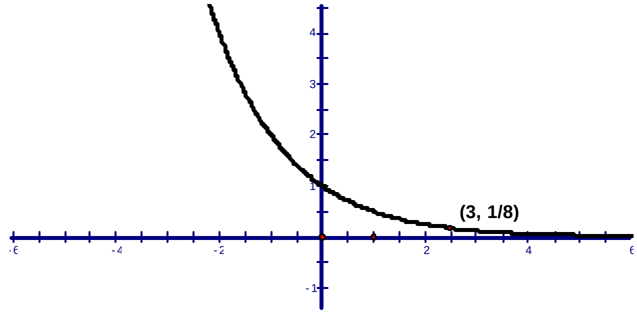
When $b > 1$, the greater the value of b is, the more ____________________________

Which function decreases more rapidly: $y = (0.5)^x$ or $y = (0.8)^x$?

When $0 < b < 1$, the smaller the value of b is, the more ____________________________

Exercise #1 Find the function $f(x) = a^x$ whose graph is given.

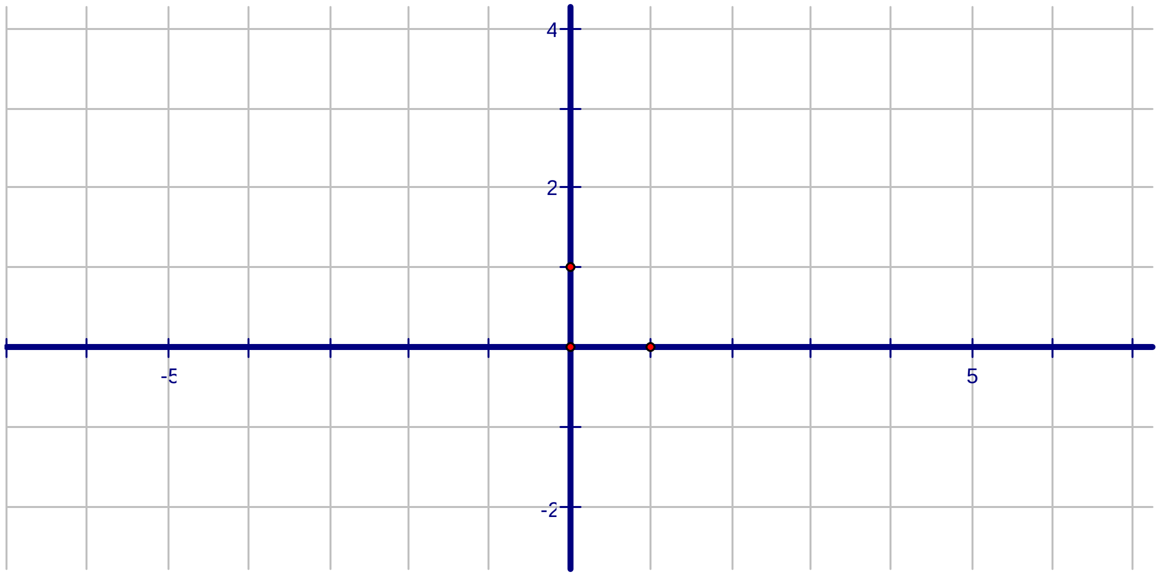




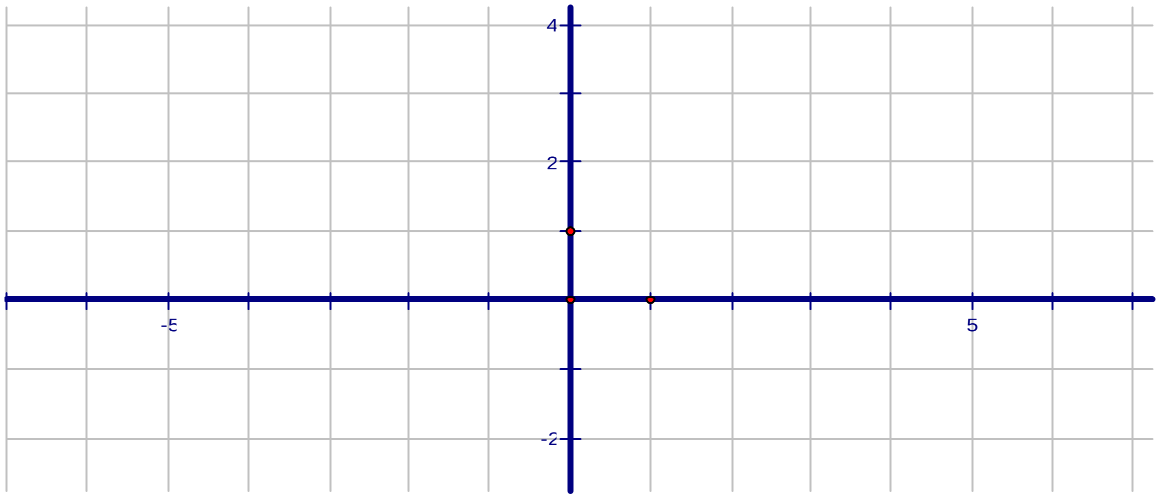
Exercise #2 Use the graph of $f(x) = 3^x$ to obtain the graph of each. Specify the domain, range, asymptote and intercept(s).

a) $g(x) = 1 + 3^x$

b) $G(x) = 3^x - 1$



c) $l(x)=3^{x+1}$ d) $L(x)=3^{x-1}$ e) $h(x)=-2^{x}$



Exercise #3 Solve the following equations:

a) $2^{x^2-2x}=8$

b) $3x\left(10^{x}\right)+10^{x}=0$

c) $4^{x+2}-4^{x}=15$

d) $x^{3}2^{x}-3\left(2^{x}\right)=0$

Exercise #4 Let $f(x) = 2^x$. Show that $\dfrac{f(x+h) - f(x)}{h} = 2^x \left(\dfrac{2^h - 1}{h} \right)$

Compound Interest $$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

A= amount in the account after t years
P = principal (amount invested)
r = annual interest rate
n = number of times interest is compounded per year
t = number of years

Exercise #5 Assume we invest \$1000 in an account that pays 6% interest rate per year.
How much is in the account at the end of one year if

a) interest is compounded once a year?

b) interest is compounded quarterly?

How much interest was paid in one year under the quarterly compounding?

What percentage of \$1000 does this represent?

________ interest paid is ________% of \$1000 .

This is called the **effective yield** (effective annual rate of interest).
The **effective yield** of an interest is the simple interest rate that would yield the same amount in 1 year.

The Number *e*

An interesting situation occurs if we consider the compound interest formula for $P = \$1$, $r = 100\%$, $t = 1$ year.

The formula becomes ______________________________.

The following table shows some values, rounded to eight decimal places, of $\left(1+\frac{1}{n}\right)^n$

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000000
10	2.5937246
100	2.70481383
1000	2.71692393
10,000	2.71814593
100,000	2.71826824
1,000,000	2.71828047
10,000,000	2.71828169
100,000,000	2.71828181
1,000,000,000	2.71828183

a) For a fixed period of time (say one year), does more and more frequent compounding of interest continue to yield greater and greater amounts?

b) Is there a limit on how much money can accumulate in a year when interest is compounded more and more frequently?

The table suggests that as n increases, the value of $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to some fixed number. The fixed number is called e. To five decimal places, $e = 2.71828$.

When $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$ and the formula for

continuously compounded interest is $\boxed{A = Pe^{rt}}$

Exercise #6 Assume we invest $1000 in an account that pays 6% interest rate per year compounded continuously.

a) How much is in the account at the end of one year?

b) How much interest was paid in one year under the continuous compounding?

c) What is the effective yield?

Exercise #7 What investment yields the greater return: 7% compounded monthly or 6.85% compounded continuously? (Hint: To answer such a question, we need to compare the effective yield of the accounts).

For the 7% compounded monthly

For the 6.85 compounded continuously

Exercise #8 (4.2 - #83)

The exponential growth of the deer population in Massachusetts can be calculated using the model $T = 50{,}000(1 + 0.06)^n$, where 50,000 is the initial deer population and 0.06 is the rate of growth. T is the total population after n years have passed.

a) Predict the total population after 4 years.

b) If the initial population was 30,000 and the growth rate was 0.12, approximately how many deer would be present after 3 years?

Exercise #9

A mobile home loses 20% of its value every 3 years.

a) A certain mobile home costs $20,000. Write a function for its value after t years.

b) How long will it be before a $20,000 mobile home depreciates to $12,800?

Exercise #10

a) Complete the table of values.
b) Write a function that describes the exponential growth.
c) Graph the function.
d) Evaluate the function at the given values.

A colony of bacteria starts with 300 organisms and doubles every week. How many bacteria will there be after 8 weeks?
After 5 days?

Weeks	0	1	2	3	4
Bacteria					