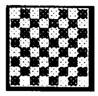
Section 4.2
Exponential Functions



The story of chess . It seems that chess had its beginnings in India around 600AD -1,400 years ago. There was a King in India who loved to play games. He commissioned a poor mathematician who lived in his kingdom to come up with a new game. After months of struggling with all kinds of ideas the mathematician came up with the game of Chaturanga. The game had two armies each lead by a King who commanded the army to defeat the other by capturing the enemy King. It was played on a simple 8x8 square board. The King loved this game so much that he offered to give the mathematician anything he wished for. ***"I would like one grain of rice for the first square of the board, two grains for the second, four grains for the third and so on doubled for each of the 64 squares of the game board"*** said the mathematician. "Is that all?" asked the King, "Why don't you ask for gold or silver coins instead of rice grains". "The rice should be sufficient for me." replied the mathematician. The King ordered his staff to lay down the grains of rice and soon learned that all the wealth in his kingdom would not be enough to buy the amount of rice needed on the 64th square. In fact the whole kingdoms supply of rice was exhausted before the 40th square was reached. "You have provided me with such a great game and yet I cannot fulfill your simple wish. You are indeed a genius." said the King and offered to make the mathematician his top most advisor instead.

Question #1: Can you find exactly how many grains of rice would be needed on the 64th square and how much total rice would be needed for all 64 squares?

1st square 1
2nd 2
3rd 2^2
4th 2^3
$\vdots$
64th 2^{63}

Total # of grains $= 1+2+2^2+\dots+2^{63}$

Let $S = 1+2+2^2+\dots+2^{63}$ $\quad /\cdot 2$

$2S = 2+2^2+2^3+\dots+2^{64}$

$2S - S = 2^{64} - 1$

$\boxed{S = 2^{64} - 1}$ $\simeq$ 18 billion billions. So if a bag of rice contained a billion grains, one would need 18 billion such bags.

Question #2: Suppose that your mathematics instructor, in an effort to improve classroom attendance, offers to pay you each day for attending class! Suppose you are to receive 2 cents on the first day you attend class, 4 cents the second day, 8 cents the third day, and so on for 30 days. What would you rather have: $1 million dollars or the above offer?

1st day 2 cents
2nd 2^2
$\vdots$
30th 2^{29}

$1+2+\dots+2^{29} = 2^{30}-1$

Total \$ $= 2^{30}-2$

$2^{30} = (2^{10})^3 \simeq (10^3)^3 = 10^9 = 1$ billion cents

Note: Simple method for quickly estimating powers of two

$$2^{10} \approx 10^3$$

The Exponential Function: $\boxed{f(x) = ab^x, \text{ where } b > 0 \text{ and } b \neq 1, a \neq 0.}$

a = the coefficient, b = base.

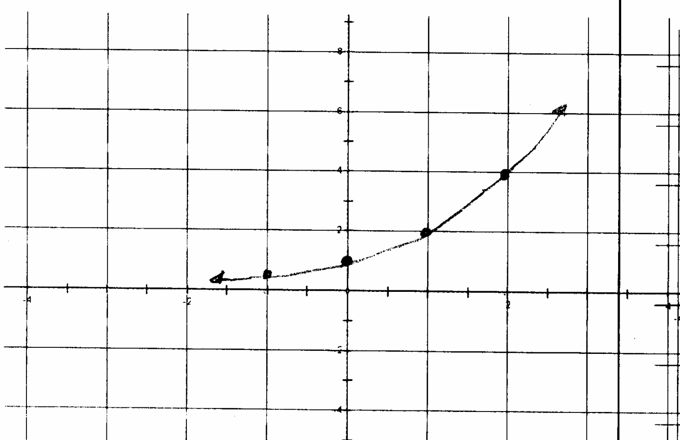
Note: If $b = 1$, then $b^x = 1^x = 1$ for any x – trivial.
If $b = 0$, then $b^x = 0^x$ which is undefined when $x = 0$
If $b < 0$, let $f(x) = (-2)^x$. If $x = \frac{1}{2}$, $f(x) = (-2)^{\frac{1}{2}} = \sqrt{-2} \notin \mathbb{R}$

Graphs of Exponential Functions

$$f(x) = 2^x$$

x	$-\infty$	-2	-1	0	1	2	3	∞
y	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	∞

$\boxed{y=0}$ H.A



Domain: $x \in \mathbb{R}$

Range: $y \in (0, \infty)$

Horizontal Asymptote: $y = 0$

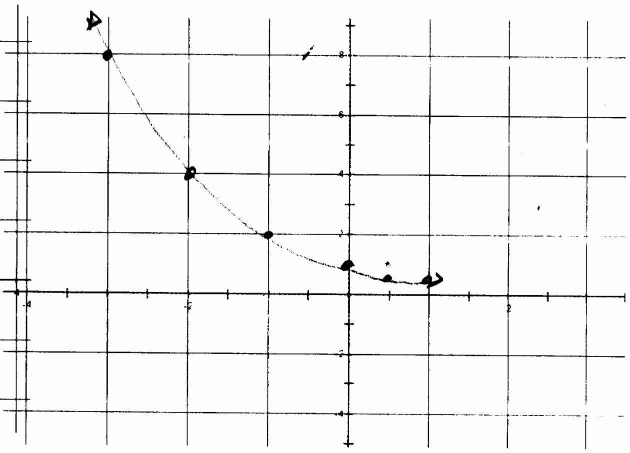
If $b > 1$, the function is **increasing**.

The function is one-to-one.

$$g(x) = \left(\frac{1}{2}\right)^x$$

x	$-\infty$	-2	-1	0	1	2	∞
y	∞	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	0

$\boxed{y=0}$ H.A



Domain: $x \in \mathbb{R}$

Range: $y \in (0, \infty)$

Horizontal Asymptote: $y = 0$

If $0 < b < 1$, the function is **decreasing.**

The function is one-to-one.

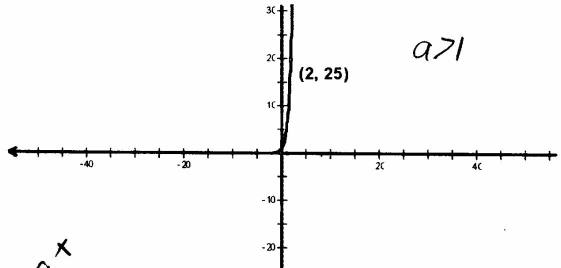
Question: Which function grows more rapidly: $y = 3^x$ or $y = 4^x$?

When $b > 1$, the greater the value of b is, the more rapidly the graph rises

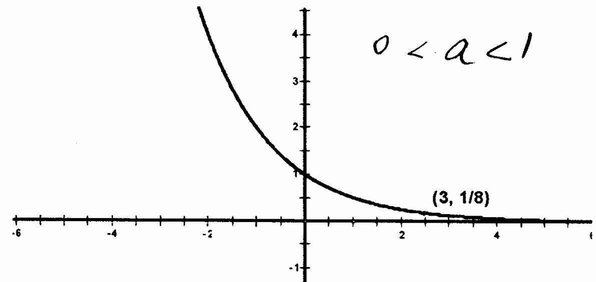
Which function decreases more rapidly: $y = (0.5)^x$ or $y = (0.8)^x$?

When $0 < b < 1$, the smaller the value of b is, the more rapidly the graph falls.

Exercise #1 Find the function $f(x)=a^x$ whose graph is given.



$a>1$



$0<a<1$

$y=a^x$

$(2,25)\in$ graph $\Rightarrow$ $25=a^2$ $\quad \sqrt{25}=\sqrt{a^2}$

$5^2=a^2$ $\quad a=\pm 5$, $a=5$ (the base >0)

$\boxed{a=5}$ $\quad$ or the exponential is one-to-one

$(3,\frac{1}{8})\in$ graph

$\frac{1}{8}=a^3 \Rightarrow a=\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$

$\boxed{a=\frac{1}{2}}$

Exercise #2 Use the graph of $f(x)=2^x$ to obtain the graph of each. Specify the domain, range, asymptote and intercept(s).

a) $g(x)=1+2^x$

shift $y=2^x$ up 1 unit

Domain: $x\in\mathbb{R}$

Range $y\in(1,\infty)$

H.A $y=1$

x-int none

y-int $(0,2)$

b) $G(x)=2^x-1$

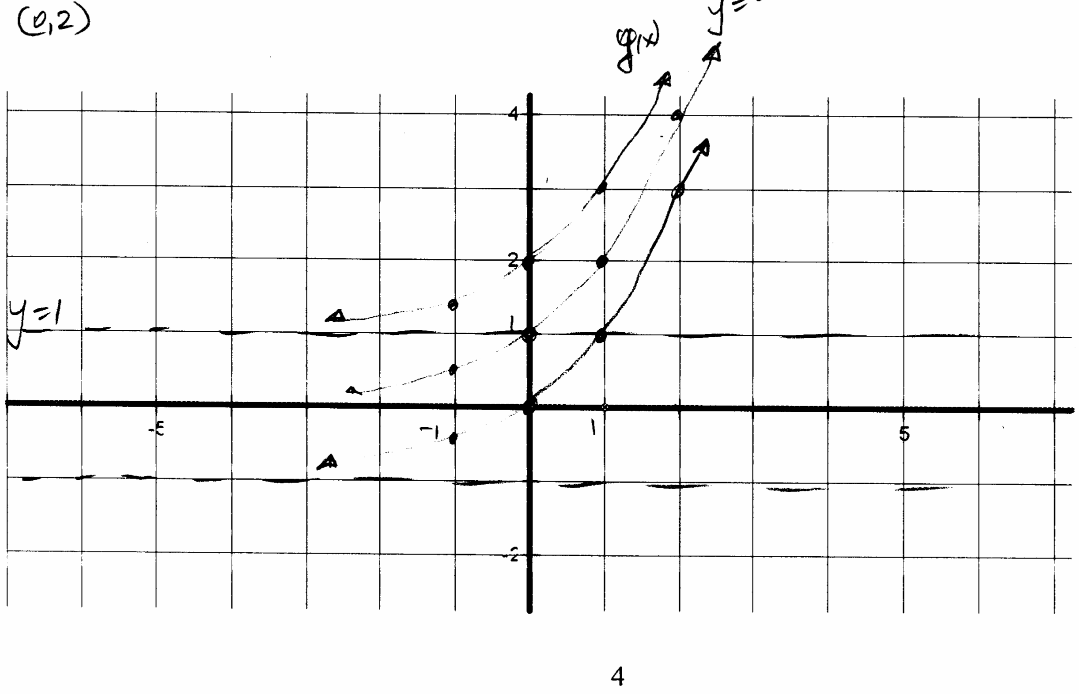
shift $y=2^x$ down 1 unit

Domain: $x\in\mathbb{R}$

Range: $y\in(-1,\infty)$

H.A: $y=-1$

x-int, y-int $(0,0)$



c) $l(x) = 2^{x+1}$

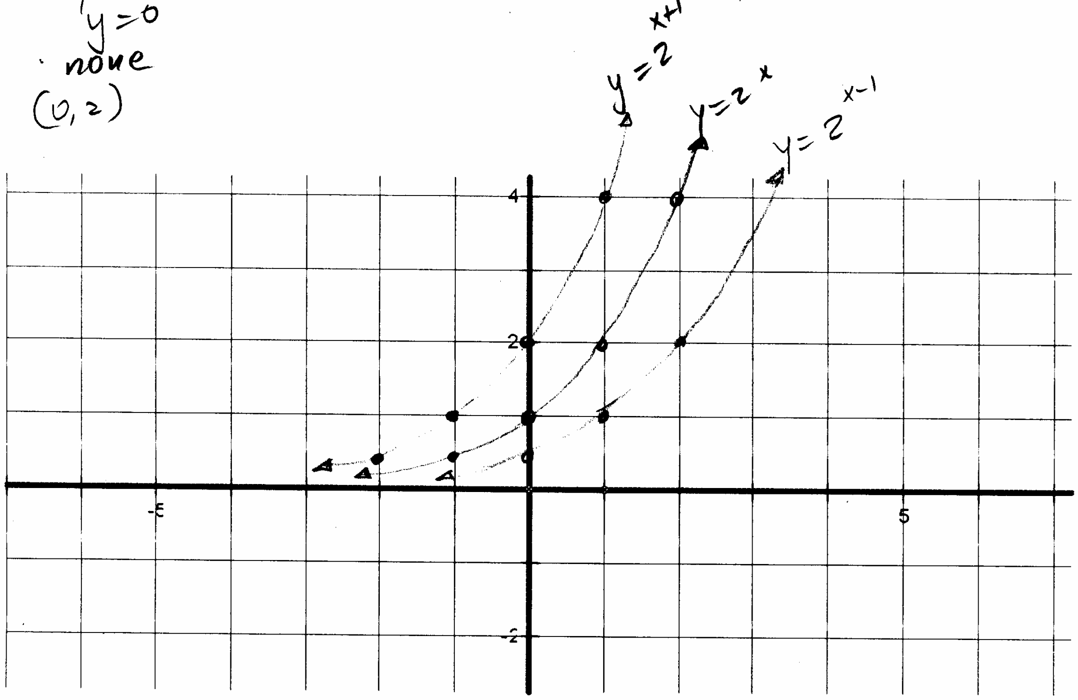
shift $y = 2^x$ left 1 unit

Domain: $x \in \mathbb{R}$
Range $y \in (0, \infty)$
H.A $y = 0$
x-int: none
y-int: $(0, 2)$

d) $L(x) = 2^{x-1}$

shift $y = 2^x$ right 1 unit

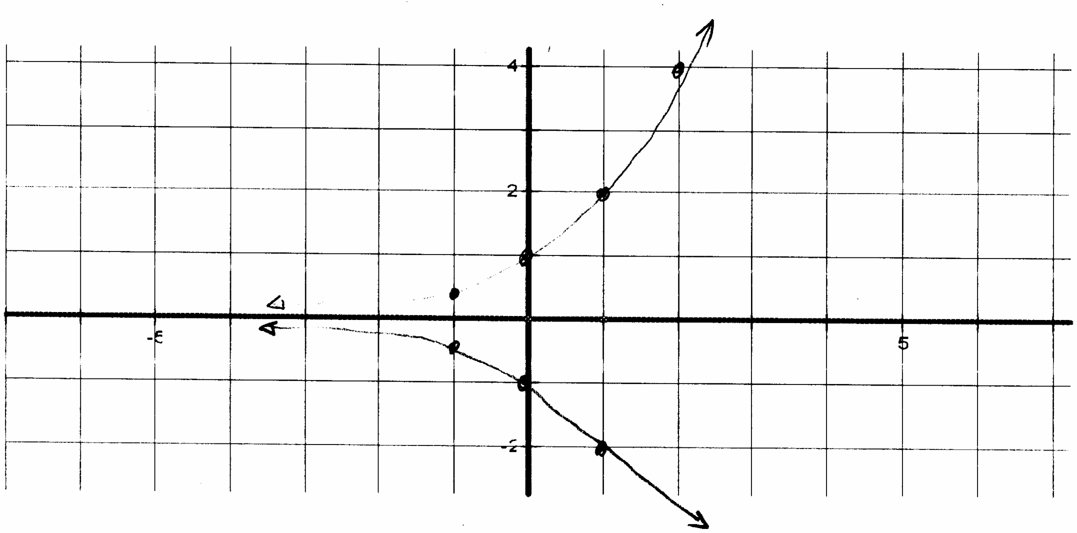
$x \in \mathbb{R}$
$y \in (0, \infty)$
H.A $y = 0$
x-int none
y-int $(0, \frac{1}{2})$



e) $h(x) = -2^x$

reflect $y = 2^x$ about the x-axis

$x \in \mathbb{R}$
$y \in (-\infty, 0)$
H.A $y = 0$
x-int none
y-int $(0, -1)$



Exercise #3 Solve the following equations:

a) $2^{x^2-2x}=8$

$2^{x^2-2x}=2^3$

$x^2-2x=3$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3$ OR $x=-1$

check: $2^3=8$

$2^3=8$

b) $3x\left(10^x\right)+10^x=0$

$10^x(3x+1)=0$

$10^x \neq 0 \Rightarrow 3x+1=0$

$x=\frac{-1}{3}$

c) $4^{x+2}-4^x=15$

$4^x \cdot 4^2-4^x=15$

$4^x(16-1)=15$

$15 \cdot 4^x=15$

$4^x=1$

$\boxed{x=0}$

d) $x^3 2^x-3\left(2^x\right)=0$

$2^x(x^3-3)=0$

$2^x \neq 0 \Rightarrow x^3-3=0$

$x^3=3$

$x=\sqrt[3]{x}$

Exercise #4 Let $f(x)=2^x$. Show that $\dfrac{f(x+h)-f(x)}{h}=2^x\left(\dfrac{2^h-1}{h}\right)$

$$\frac{f(x+h)-f(x)}{h}=\frac{2^{x+h}-2^x}{h}=\frac{2^x \cdot 2^h-2^x}{h}=\frac{2^x(2^h-1)}{h}=2^x \cdot\left(\frac{2^h-1}{h}\right)$$

Compound Interest

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A= amount in the account after t years
P = principal (amount invested)
r = annual interest rate
n = number of times interest is compounded per year
t = number of years

Exercise #5 Assume we invest $1000 in an account that pays 6% interest rate per year. How much is in the account at the end of one year if

a) interest is compounded once a year?

$P = 1000$
$r = 0.06$
$n = 1$
$t = 1$

$A = 1000\left(1+\frac{0.06}{1}\right)^{1}$

$A = 1000\,(1.06) = 1060$ $

b) interest is compounded quarterly?

$P = 1000$
$r = 0.06$
$n = 4$
$t = 1$

$A = 1000\left(1+\frac{0.06}{4}\right)^{4\cdot 1}$

$A = 1000\,(1.0614)$

$A = 1061.4$ $

How much interest was paid in one year under the quarterly compounding?

$1061.40 - 1000 = 61.40$ $ interest paid

What percentage of $1000 does this represent?

$ 61.40 interest paid is 6.14 % of $1000.

This is called the **effective yield** (effective annual rate of interest).
The **effective yield** of an interest is the simple interest rate that would yield the same amount in 1 year.

It shows that the initial amount 1000 would grow to $1000(1.0614)$ after 1 year.

The Number e

An interesting situation occurs if we consider the compound interest formula for $P = \$1$, $r = 100\%$, $t = 1$ year.

The formula becomes $A = \left(1+\frac{1}{n}\right)^n$.

The following table shows some values, rounded to eight decimal places, of $\left(1+\frac{1}{n}\right)^n$

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000000
10	2.5937246
100	2.70481383
1000	2.71692393
10,000	2.71814593
100,000	2.71826824
1,000,000	2.71828047
10,000,000	2.71828169
100,000,000	2.71828181
1,000,000,000	2.71828183

a) For a fixed period of time (say one year), does more and more frequent compounding of interest continue to yield greater and greater amounts?

yes

b) Is there a limit on how much money can accumulate in a year when interest is compounded more and more frequently?

yes

The table suggests that as n increases, the value of $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to some fixed number. The fixed number is called e. To five decimal places, $e = 2.71828$.

When $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$ and the formula for **continuously compounded interest** is $\boxed{A = Pe^{rt}}$

Exercise #6 Assume we invest \$1000 in an account that pays 6% interest rate per year compounded continuously.

a) How much is in the account at the end of one year?

$A = 1000\, e^{0.06} = 1061.84$

b) How much interest was paid in one year under the continuous compounding?

\$ 61.84

c) What is the effective yield? = simple interest that would yield the same amount 1061.84 after 1 year

$61.84 = x\%$ of 1000

$61.84 = 6.184\%$ = effective yield

Exercise #7 What investment yields the greater return: 7% compounded monthly or 6.85% compounded continuously? (Hint: To answer such a question, we need to compare the effective yield of the accounts).

For the 7% compounded monthly

Let R = effective yield

$\cancel{P}(1+R) = \cancel{P}\left(1+\frac{r}{n}\right)^{nt}$

simple interest   compounded

$1+R = \left(1+\frac{0.07}{12}\right)^{12}$

$R = 1.07186 - 1$

$R = 0.07186 = 7.186\%$

For the 6.85 compounded continuously

R = effective yield

$\cancel{P}(1+R) = \cancel{P} \cdot e^{rt}$

$1+R = e^{0.0685 \cdot 1}$

$1+R = 1.07090$

$R = 0.0709 = 7.09\%$

It is better to invest 7% compounded mo

Exercise #8 (4.2 - #75)

The exponential growth of the deer population in Massachusetts can be calculated using the model $T = 50{,}000(1+0.06)^n$, where 50,000 is the initial deer population and 0.06 is the rate of growth. T is the total population after n years have passed.

a) Predict the total population after 4 years.

$n = 4$

$T = 50{,}000(1+0.06)^4$

$T \approx 63{,}123$ deer ✓

b) If the initial population was 30,000 and the growth rate was 0.12, approximately how many deer would be present after 3 years?

$T = 30{,}000(1+0.12)^n$

$n = 3$

$T = 42{,}147$ deer ✓

Exercise #9 A mobile home loses 20% of its value every 3 years.

a) A certain mobile home costs $20,000. Write a function for its value after t years.

t	V
0	20,000
3	80%(20,000) = 0.8(20,000)
6	$0.8(0.8)(20{,}000) = (0.8)^2\, 20{,}000$
9	$(0.8)^3(20{,}000)$

$\boxed{V = 20{,}000(0.8)^{t/3}}$

b) How long will it be before a $20,000 mobile home depreciates to $12,800?

$V = 12{,}800$

$12{,}800 = 20{,}000(0.8)^{\frac{t}{3}}$

$0.64 = (0.8)^{t/3}$

$(0.8)^2 = (0.8)^{t/3}$

$\frac{t}{3} = 2 \Rightarrow \boxed{t = 6}$

Exercise #10

a) Complete the table of values.
b) Write a function that describes the exponential growth.
c) Graph the function.
d) Evaluate the function at the given values.

A colony of bacteria starts with 300 organisms and doubles every week. How many bacteria will there be after 8 weeks?
After 5 days?

Weeks	0	1	2	3	4
Bacteria	300	600	1200	2400	4800

w	B
0	300
1	2(300)
2	$2(2(300)) = 2^2(300)$
3	$2^3(300)$
w	$2^w(300)$

$\boxed{B(w) = 300(2^w)}$

$w = 8$, $B(8) = 300(2^8) = \boxed{76{,}800}$

5 days $= \frac{5}{7}$

$B(\frac{5}{7}) = 300(2^{\frac{5}{7}}) = \boxed{492}$