3.5 Graphs of Rational Functions

Example 1 Graph the reciprocal function $\boxed{f(x)=\frac{1}{x}}$ Answer the following questions:

a) What is the domain of the function? $x \in \mathbb{R} \setminus \{0\}$

b) What is the range of the function? $y \in \mathbb{R} \setminus \{0\}$

c) What are the x-and y-intercepts? none

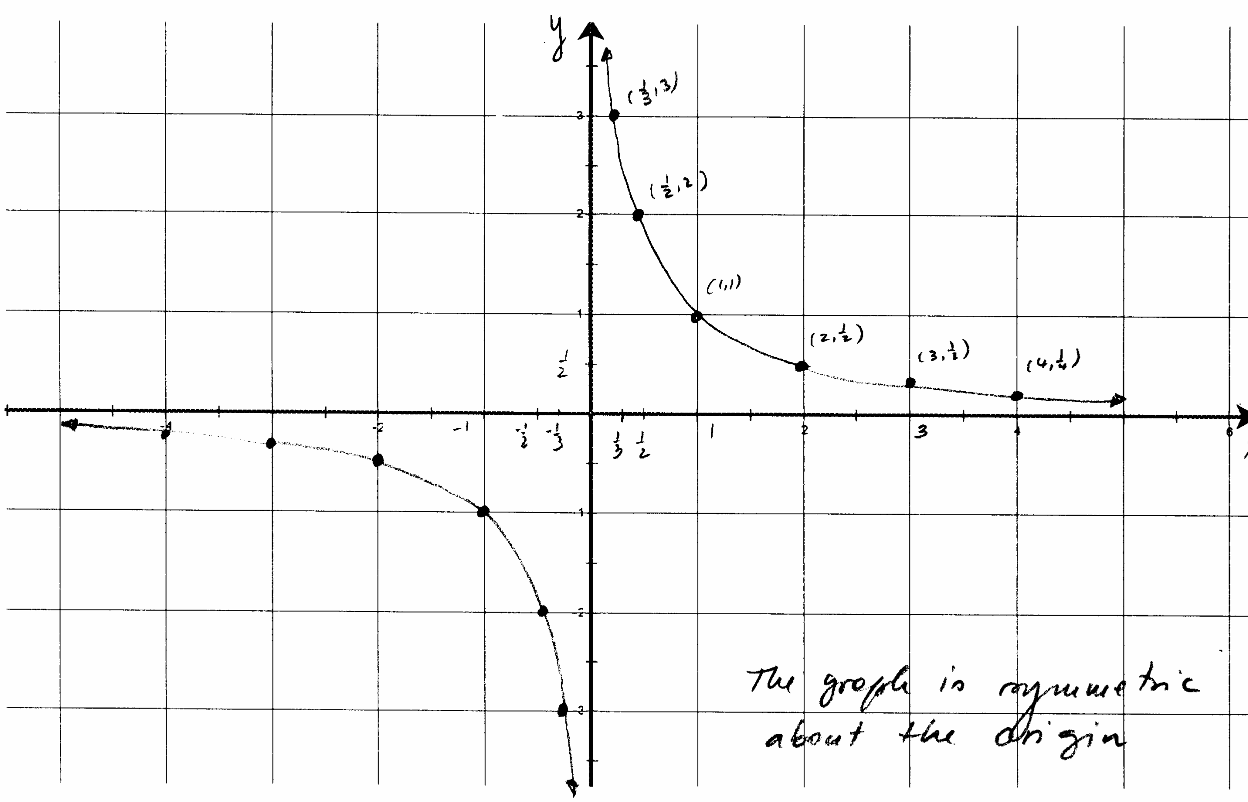
d) What is the end- behavior of the function, that is, what happens with the values of y as x goes to ∞ and $-\infty$?

when $x \to \infty$, $f(x) \to 0$; when $x \to -\infty$, $f(x) \to 0$

e) What is the behavior of the function when x approaches 0 ?

when $x \to 0^+$, $f(x) \to \infty$; when $x \to 0^-$, $f(x) \to -\infty$

x	$-\infty$	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4	∞
$f(x)$	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	$-\infty \mid \infty$	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	0
	$\boxed{y=0}$ H.A						$\boxed{x=0}$ V.A						$\boxed{y=0}$ H.A



Definition A **rational function** is a function *f* of the form $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x)\neq 0$.

Notations:

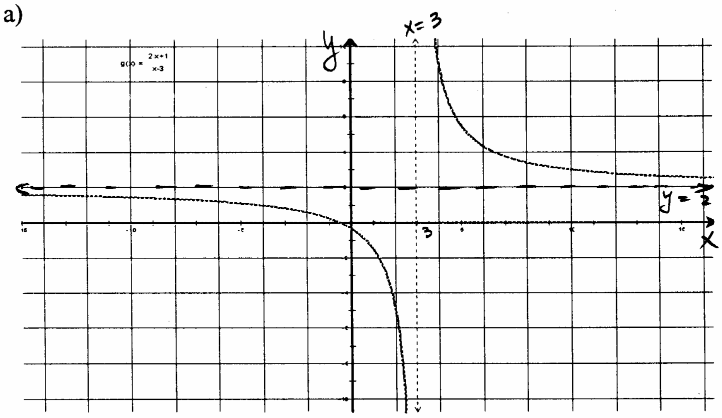
$x\to\infty$	*x* approaches infinity (x increases without bound)
$x\to-\infty$	*x* approaches negative infinity (*x* decreases without bound)
$x\to a^{+}$	*x* approaches a from the right
$x\to a^{-}$	*x* approaches a from the left

Definition The line $\boxed{x=a}$ is a **vertical asymptote** for the graph of $f(x)$ if, when $x\to a$, $y\to\pm\infty$.

The line $\boxed{y=b}$ is a **horizontal asymptote** for the graph of $f(x)$ if, when $x\to\pm\infty$, $y\to b$.

Exercise #1 Identify all the vertical and horizontal asymptotes of the following graphs.
How can the vertical asymptotes be found? What about the horizontal asymptotes?

a)



$$f(x)=\frac{2x+1}{x-3}$$

V.A. $\boxed{x=3}$ because when $x\to 3$, $y\to\pm\infty$

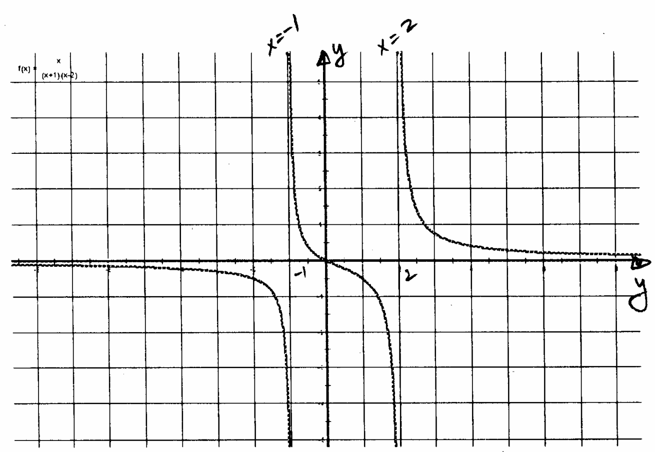
$x=3$ is a zero of the denominator.

$f(x)$ is not defined in $x=3$

H.A. $\boxed{y=2}$ because when $x\to\pm\infty$, $y\to 2$

degree numerator = degree of denominat

b)



$$f(x)=\frac{x}{(x+1)(x-2)}$$

V.A $\boxed{x=2}$ because when $x\to 2$, $y\to\pm\infty$

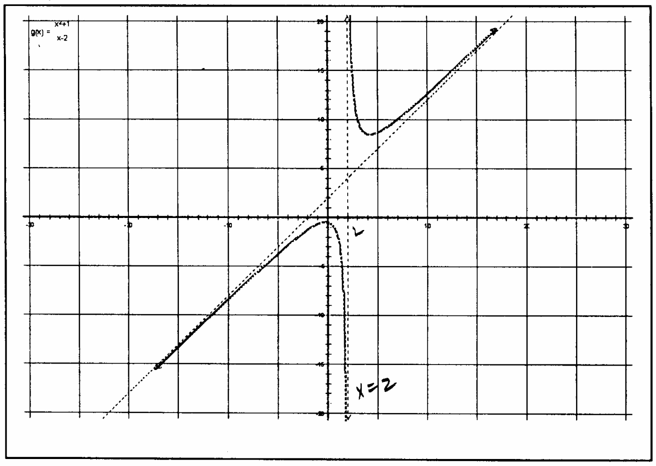
$\boxed{x=-1}$ because when $x\to -1$, $y\to\pm\infty$

$x=2$ and $x=-1$ are zeros of the denominator

H.A. $\boxed{y=0}$ because when $x\to\pm\infty$, $y\to 0$

degree numerator < degree denominator

c)



$$f(x)=\frac{x^2+1}{x-2}$$

V.A $\boxed{x=2}$ because when $x \to 2, y \to \pm\infty$

$x=2$ is a zero of the denominator

H.A – none

degree numerator > degree denominator

Oblique asymptote

when $x \to \pm\infty$, $y \to$ oblique asymptote

Asymptotes for a rational function $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_n x^n+\ldots+a_0}{b_m x^m+\ldots b_0}$

1. The vertical asymptotes are the lines $x=c$, where c is a zero of the denominator.

2. If $n<m$, then $y=0$ (the x-axis) is the **horizontal asymptote.**

If $n=m$, then $y=\dfrac{a_n}{b_n}$ is the **horizontal asymptote.**

If $n>m$, there are **no horizontal asymptotes.**

If, however, $n=m+1$, then there is an oblique asymptote . Divide the numerator by the denominator and disregard the remainder.

y = quotient is the oblique asymptote

Exercise #2 Identify all the asymptotes for the following functions:

$f(x)=\dfrac{2x+7}{x-5}$

V.A. $x=5$

H.A $y=\frac{2}{1}=2$

no oblique asymptote

$g(x)=\dfrac{4x^2+x-5}{2x^2-3x-5}$

$g(x)=\dfrac{(4x+5)(x-1)}{(2x-5)(x+1)}$

V.A $x=\frac{5}{2}$ and $x=-1$

H.A. $y=\frac{4}{2}=2$

no oblique asymptote

$h(x)=\dfrac{x^2+6}{x-3}$

V.A $x=3$

Oblique asymptote

$$\begin{array}{r|l} & x+3 \\ x-3 & x^2+6 \\ & -x^2+3x \\ & 3x+6 \\ & -3x+9 \\ & 15 \end{array}$$

$y=x+3$

$l(x)=\dfrac{1}{2x^2-2}$

$l(x)=\dfrac{1}{2(x+1)(x-1)}$

V.A. $x=-1$ and $x=1$

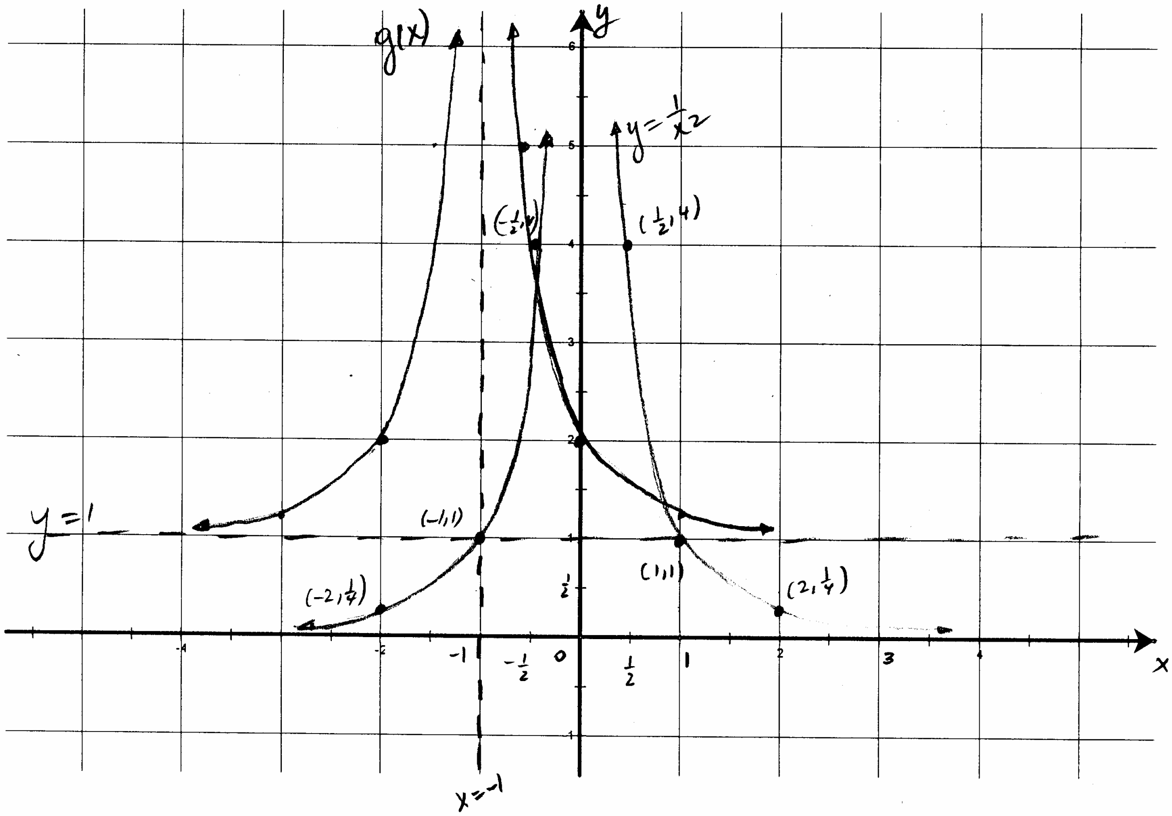
H.A. $y=0$

no oblique asymptote

Exercise #3 Graph the function $f(x)=\frac{1}{x^2}$. Find the domain, the asymptotes, and the x- and y-intercepts.

x	$-\infty$	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	∞
$f(x)$	0	$\frac{1}{9}$	$\frac{1}{4}$	1	$\frac{1}{4}$	$\infty \mid \infty$	4	1	$\frac{1}{4}$	$\frac{1}{9}$	0
	$\boxed{y=0}$ H.A					$\boxed{x=0}$ V.A					$\boxed{y=0}$ H.A

no x-intercept
no y-intercept
graph is symmetric about the y-axis



Exercise #4 Show how to obtain the graph of $g(x)=\frac{1}{(x+1)^2}+1$ from the graph of $f(x)=\frac{1}{x^2}$.

What are the asymptotes of $g(x)$?

g(x)

1st start with $y=\frac{1}{x^2}$

2nd $y=\frac{1}{(x+1)^2}$ shift left 1 unit

3rd $y=\frac{1}{(x+1)^2}+1$ shift up 1 unit

V.A. $x=-1$

H.A. $y=1$

Exercise #5 Sketch the graph of $f(x) = \dfrac{x+1}{x-4}$. Find the domain, all the asymptotes, the x- and y-intercepts. Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	$-\infty$	-2	-1	0	1	2	4	5	8	∞
y	1	$-\frac{1}{6}$	0	$-\frac{1}{4}$	$-\frac{2}{3}$	$-\frac{3}{2}$	$-\infty \mid \infty$	6	$\frac{9}{2}$	1
	$\boxed{y=1}$ H.A						$\boxed{x=4}$ V.A			$\boxed{y=1}$ H.A

x-n: $y=0 \Rightarrow x+1=0 \Rightarrow x=-1$

y-n: $x=0 \Rightarrow y=-\frac{1}{4}$

H.A $y=\frac{1}{1}=1$ $\quad f(x)=\dfrac{x(1+\frac{1}{x})}{x(1-\frac{4}{x})}=\dfrac{1+\frac{1}{x}}{1-\frac{4}{x}} \rightarrow 1$ when $x\rightarrow\pm\infty$

intersection of $f(x)$ with $y=1$

$\dfrac{x+1}{x-4}=1 \iff x+1=x-4$

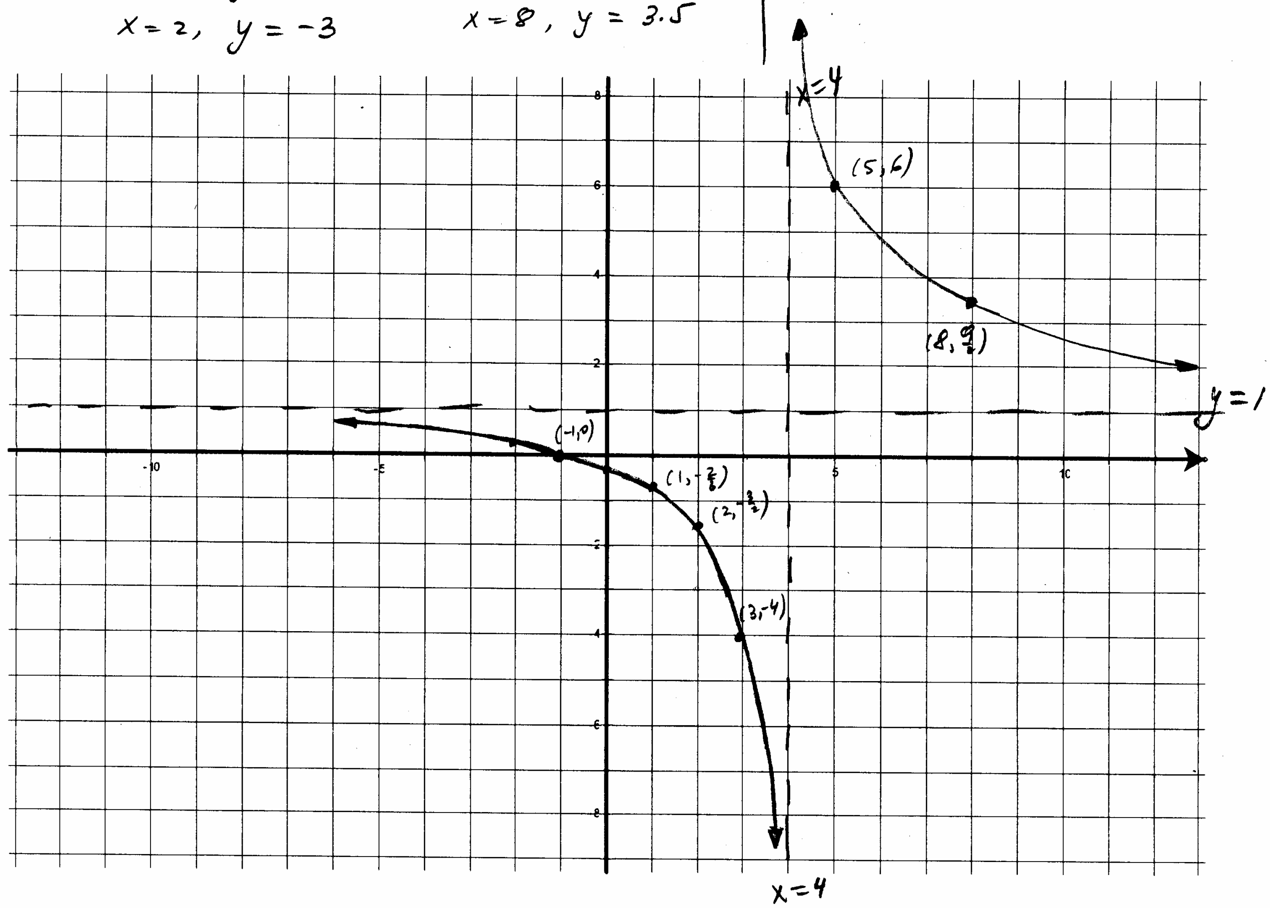
No solutions.

when $x\rightarrow 4^+$, $y\rightarrow \frac{5}{+0}=\infty$

$x\rightarrow 4^-$, $y\rightarrow \frac{5}{-0}=-\infty$

Test points $x=1$, $y=\frac{-2}{3}$ $\quad x=5$, $y=6$

$x=2$, $y=-3$ $\quad x=8$, $y=3.5$



Exercise #6 Sketch the graph of $f(x)=\dfrac{x-2}{x^2-1}$. Find the domain, all the asymptotes, the x- and y-intercepts. Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	$-\infty$	-2	-1	0	1	2	3	∞
$f(x)$	0	$-\frac{4}{3}$	$-\infty \mid \infty$	2	$\infty \mid -\infty$	0	$\frac{1}{8}$	0
	$\boxed{y=0}$ H.A.		$\boxed{x=-1}$ V.A		$\boxed{x=1}$ V.A			$\boxed{y=0}$ H.A

$f(x)=\dfrac{x-2}{(x+1)(x-1)}$ $\quad x\neq 1, x\neq -1$

H.A. $y=0$

x-n: $y=0 \iff x-2=0 \iff x=2$

y-n: $x=0 \Rightarrow y=\dfrac{-2}{-1}=2$

$x\to -1^-,\ y\to \dfrac{-3}{(-0)(-2)}=-\infty$

$x\to -1^+,\ y\to \dfrac{-3}{(+0)(-2)}=\infty$

$x\to 1^-,\ y\to \dfrac{-1}{2(-0)}=\infty$

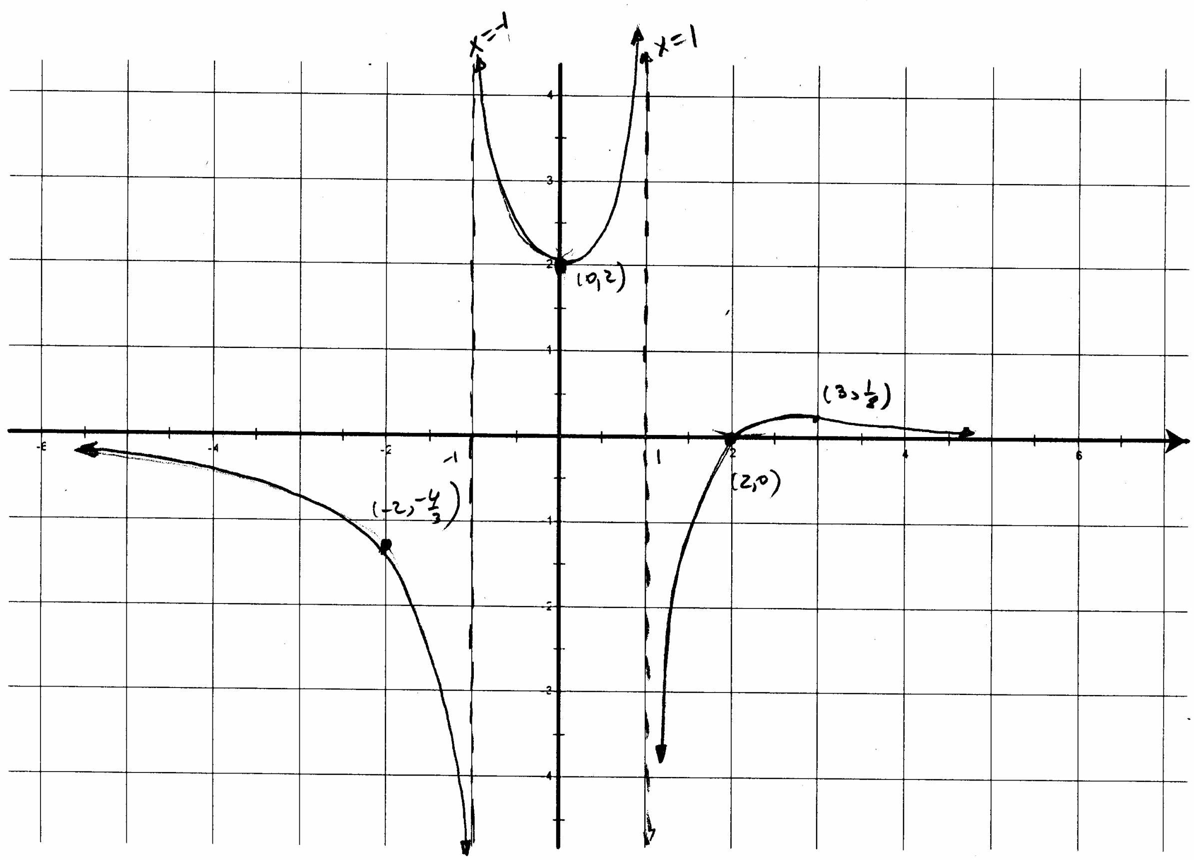
$x\to 1^+,\ y\to \dfrac{-1}{2(+0)}=-\infty$

intersection with $y=0$ $\quad \dfrac{x-2}{x^2-1}=0 \iff x=2$

Test points:

$x=3,\ y=\dfrac{1}{8}$

$x=-2,\ y=\dfrac{-4}{3}$



Exercise #7 Sketch the graph of $f(x)=\dfrac{x^2-2x-8}{x^2-4x+3}$. Find the domain, all the asymptotes, the x- and y-intercepts Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

$$f(x)=\frac{(x-4)(x+2)}{(x-3)(x-1)} \qquad x\neq 3,\ x\neq 1$$

x	$-\infty$	-2	-1	0	1	2	3	4	5.5	∞
$f(x)$	1	0		$-\frac{8}{3}$	$-\infty \mid \infty$	8	$\infty \mid -\infty$	0	1	1

$\boxed{y=1}$ H.A $\quad\boxed{x=1}$ V.A $\quad\boxed{x=3}$ V.A $\quad\boxed{y=1}$ H.A

H.A. $y=\frac{1}{1}=1$

x-n: $y=0 \Rightarrow x=4,\ x=-2$

y-n: $x=0,\ y=\frac{-8}{3}\simeq -2.6$

$x\to 1^-,\ y\to \frac{(-3)(3)}{(-2)(-0)}=-\infty$

$x\to 1^+,\ y\to \frac{(-3)3}{(-2)(+0)}=+\infty$

$x\to 3^-,\ y\to \frac{(-1)(5)}{(-0)(2)}=\infty$

$x\to 3^+,\ y\to \frac{(-1)5}{(+0)2}=-\infty$

Test points

$x=6,\ y=\frac{2\cdot 8}{3\cdot 5}=\frac{16}{15}\simeq 1.06$

$x=2,\ y=\frac{(-2)4}{(-1)1}=8$

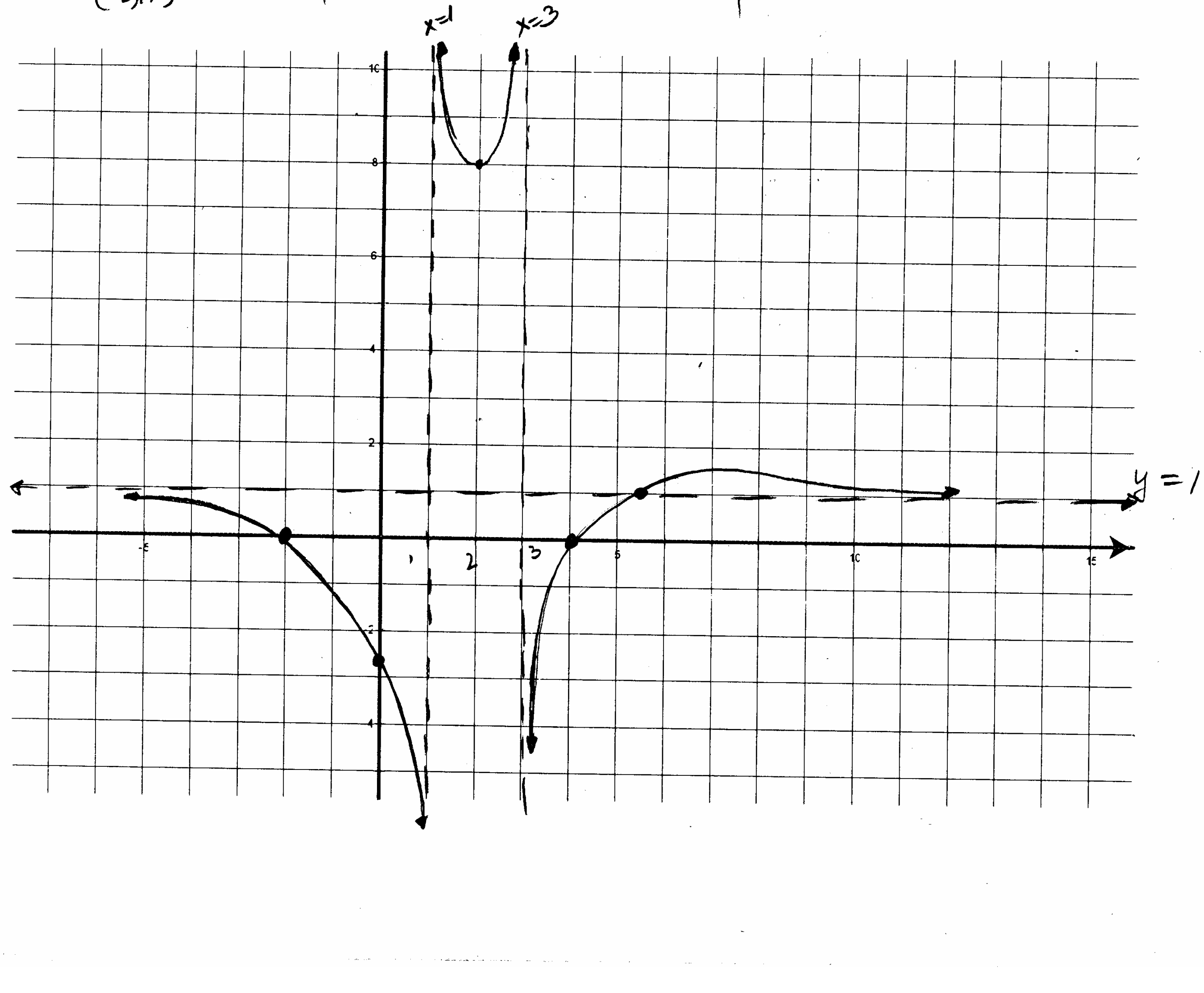
intersection with $y=1$

$\frac{x^2-2x-8}{x^2-4x+3}=1 \Rightarrow$

$\cancel{x^2}-2x-8=\cancel{x^2}-4x+3$

$2x=11$

$x=\frac{11}{2}$ $\quad (5.5,\ 1)$



Exercise #8 Sketch the graph of $f(x)=\dfrac{x^2+1}{x+3}$. Find the domain, all the asymptotes, the x- and y-intercepts. Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	$-\infty$	-4	-3	-2	-1	0	1	2	∞
	$\boxed{y=x-3}$ O.A	-9	$-\infty \mid +\infty$	5	1	$\frac{1}{3}$	$\frac{1}{2}$	1	$\boxed{y=x-3}$ O.A.

$\boxed{x=-3}$ V.A

$x \neq -3$

no horizontal asymptote

Oblique asymptote

$$\begin{array}{r} x-3 \\ x+3 \,\overline{)\, x^2+1} \\ \underline{-x^2-3x} \\ -3x+1 \end{array}$$

$\boxed{y=x-3}$

x	y
0	-3
3	0

x-n: $y=0$ no x-n

y-n: $x=0$, $y=\frac{1}{3}$

$x \to -3^-$, $y \to \frac{+10}{-0}=-\infty$

$x \to -3^+$, $y \to \frac{+10}{+0}=+\infty$

Test points

$x=2$, $y=1$

$x=-1$, $y=\frac{2}{2}=1$

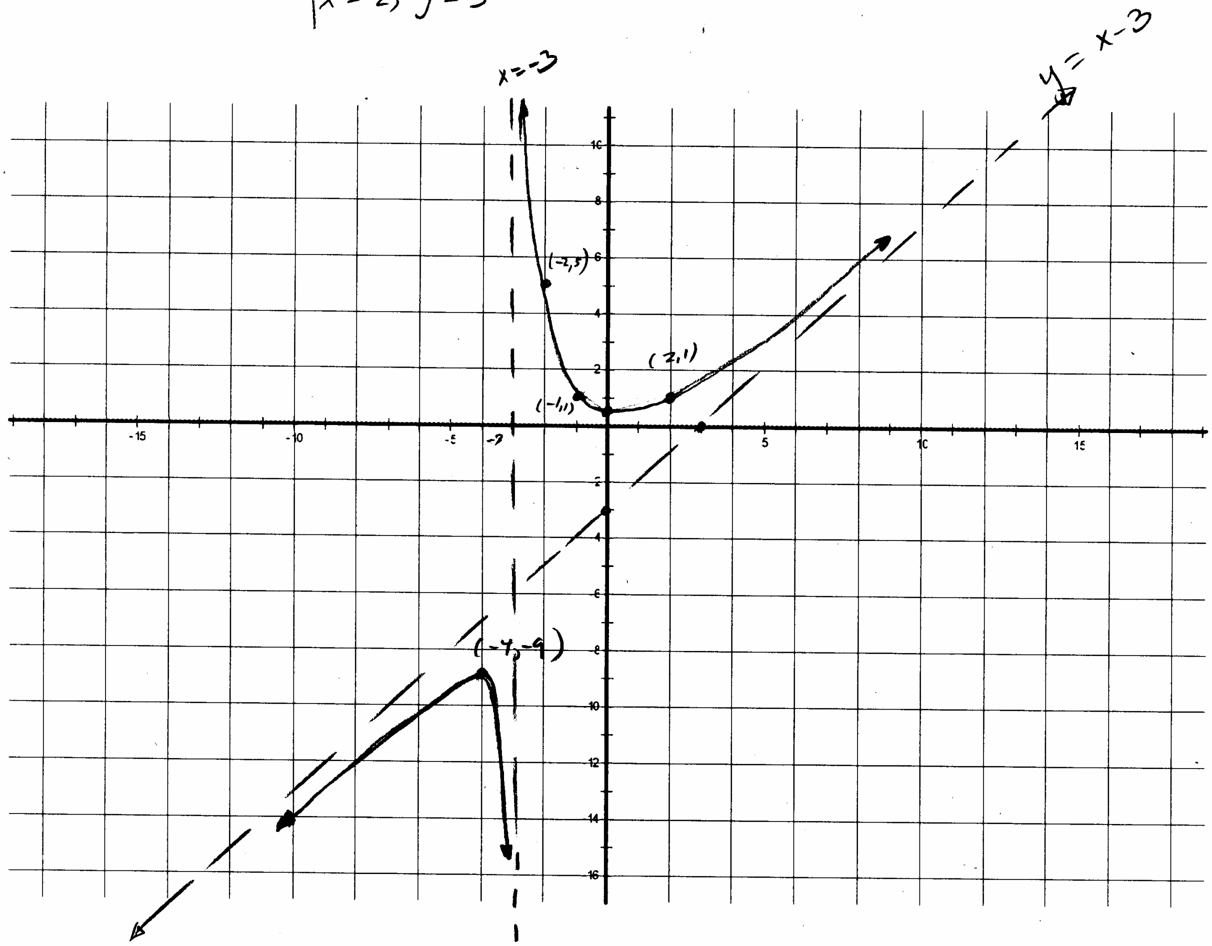
$x=-2$, $y=5$

$x=-4$, $y=-9$

intersection with $y=x-3$

$\frac{x^2+1}{x+3}=x-3 \iff x^2+1=x^2-9$

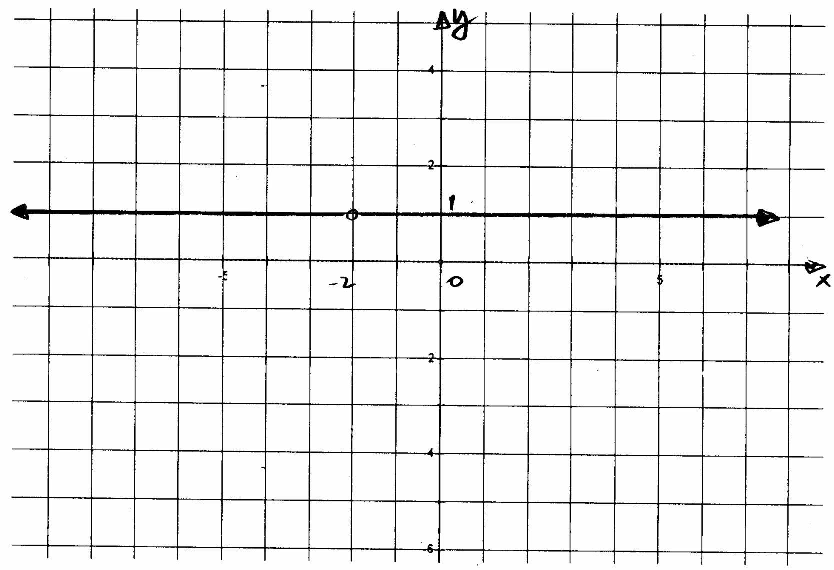
no solution.



Exercise #9 Graph the following functions: $f(x)=\dfrac{x+2}{x+2}$ and $g(x)=\dfrac{x^2-9}{x+3}$.

$f(x)=\dfrac{x+2}{x+2}=1$

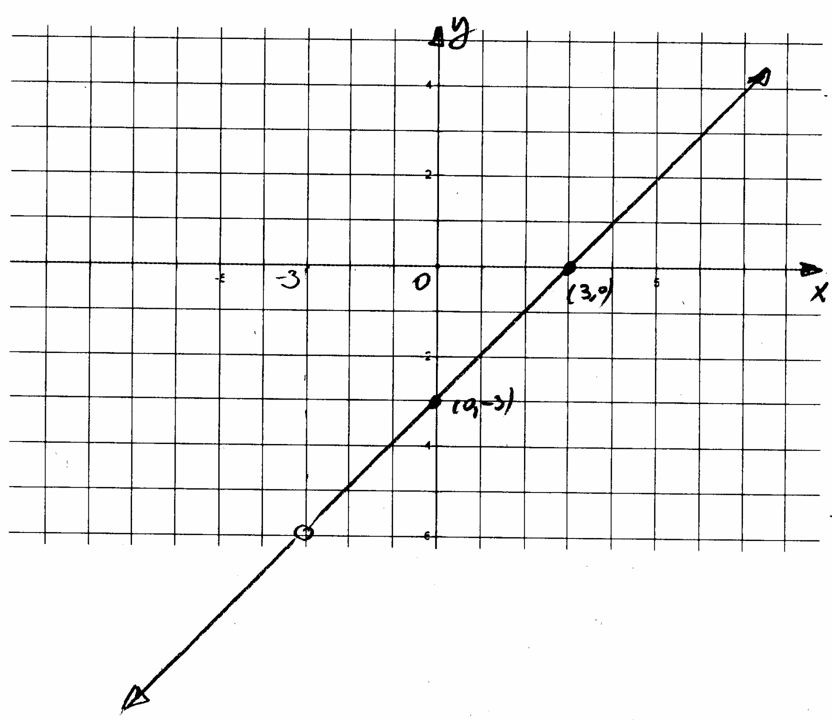
$x\neq -2$



$g(x)=\dfrac{x^2-9}{x+3}=\dfrac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}$

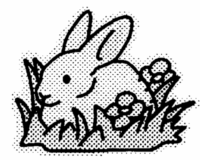
$x\neq -3$ $\boxed{g(x)=x-3}$

x	y
0	-3
3	0



3.5 Graphs of Rational Functions - Applications

1. The rabbit population on Mr. Jenkins's farm follows the formula



$$p(t) = \frac{3000t}{t+1}$$

where $t \geq 0$ is the time (in months) since the beginning of the year.

a) Sketch a graph of the rabbit population.

b) What eventually happens to the rabbit population?

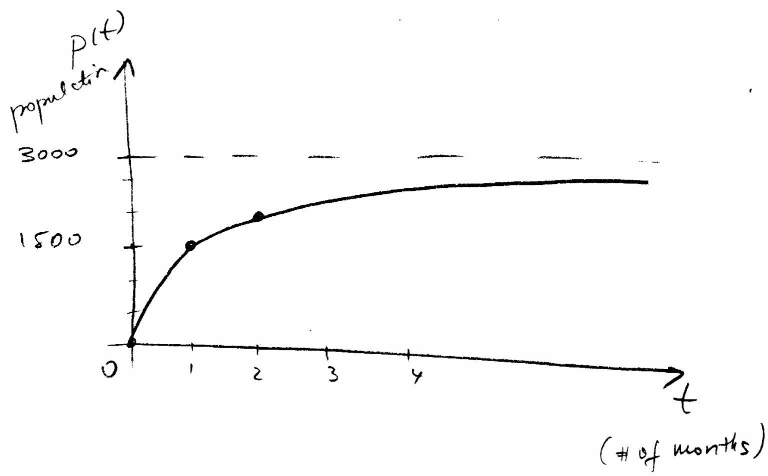
t	0	1	2	3	∞
$p(t)$	0	1500	2000	2250	$p = 3000$

VA = $t = \cancel{-1}$ no meaning.
H.A $p = 3000$

$t = 1$, $p(1) = \frac{3000}{2}$

$t = 2$, $p(2) = \frac{3000 \cdot 2}{3}$

$t = 3$, $p(3) = \frac{3000 \cdot 3}{4}$
$= 750 \cdot 3$
$= 2250$



The population levels off

2. Using rational functions to model bacterial growth

A group of agricultural scientists has been studying how the growth of a particular type of bacteria is affected by the acidity level of the soil. One colony of the bacteria is placed in a soil that is slightly acidic. A second colony of the same size is placed in a neutral soil. Suppose that after analyzing the data, the scientists determine that the size of each population over time can be modeled by the following functions.

Colony of neutral soil: $y = \dfrac{2t+1}{t+1}, t \geq 0$

t = # of hours
y = population (in thousands)

Colony of acidic soil: $y = \dfrac{4t+3}{t^2+3}, t \geq 0$

In both cases, y represents the population, in thousands, after t hours.

a) With how many bacteria does each colony begin?

$t=0$, $y_1 = 1$, $y_2 = 1$ Each colony begins with 1000 bacteria

b) Determine the long-term behavior of each colony.

$y = \dfrac{2t+1}{t+1}$ $y = \dfrac{4t+3}{t^2+3}$

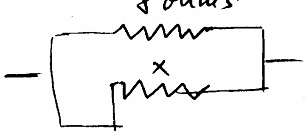
H.A. $y = 2$ H.A. $y = 0$

in the long run, this colony approaches 2000 - approaches extinction (it becomes extinct)

3. Electrical Resistance

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$



Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor. If the resistance of the variable resistor is denoted by x, then the combined resistance R is a function of x. Graph R and give a physical interpretation of the graph.

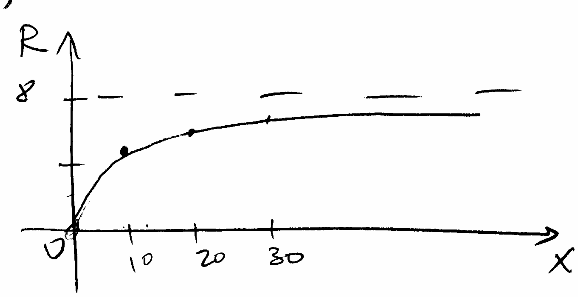
$R = \dfrac{8x}{8+x}$

Since resistance can't be negative, $x > 0$

No V.A when $x > 0$

H.A. $R = 8$

x	0	10	20	30	∞
R	1	4.4	5.7	6.3	8



R increases as x increases.

For large x, R levels off

No matter how large x is, the combined resistance is never greater than 8 ohms.