6.5 Solving Quadratic Equations by Factoring

OBJECTIVES

1. Solve quadratic equations by factoring.
2. Solve other equations by factoring.

Galileo Galilei (1564–1642) developed theories to explain physical phenomena and set up experiments to test his ideas. According to legend, Galileo dropped objects of different weights from the Leaning Tower of Pisa to disprove the belief that heavier objects fall faster than lighter objects. He developed a formula for freely falling objects described by

$$d = 16t^2,$$

where d is the distance in feet that an object falls (disregarding air resistance) in t seconds, regardless of weight.

The equation $d = 16t^2$ is a *quadratic equation*, the subject of this section. A quadratic equation contains a squared term and no terms of higher degree.

Quadratic Equation

A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$.

AUDIO

The form $ax^2 + bx + c = 0$ is the **standard form** of a quadratic equation. For example,

$$x^2 + 5x + 6 = 0, \qquad 2a^2 - 5a = 3, \qquad \text{and} \qquad y^2 = 4$$

are all quadratic equations, but only $x^2 + 5x + 6 = 0$ is in standard form.

Up to now, we have factored *expressions*, including many quadratic expressions of the form $ax^2 + bx + c$. In this section, we see how we can use factored quadratic expressions to solve quadratic *equations*.

OBJECTIVE 1 Solve quadratic equations by factoring. We use the **zero-factor property** to solve a quadratic equation by factoring.

EXPLORE AUDIO

Zero-Factor Property

If a and b are real numbers and if $ab = 0$, then $a = 0$ or $b = 0$.

That is, if the product of two numbers is 0, then at least one of the numbers must be 0. One number *must* be 0, but both *may* be 0.

YOU TRY IT

EXAMPLE 1 Using the Zero-Factor Property

Solve each equation.

(a) $(x + 3)(2x - 1) = 0$

The product $(x + 3)(2x - 1)$ is equal to 0. By the zero-factor property, the only way that the product of these two factors can be 0 is if at least one of the factors equals 0. Therefore, either $x + 3 = 0$ or $2x - 1 = 0$. Solve each of these two linear equations as in Chapter 2.

$$x + 3 = 0 \quad \text{or} \quad 2x - 1 = 0 \qquad \text{Zero-factor property}$$

$$x = -3 \qquad 2x = 1 \qquad \text{Add 1 to each side.}$$

$$x = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

The given equation, $(x + 3)(2x - 1) = 0$, has two solutions, -3 and $\frac{1}{2}$. Check these solutions by substituting -3 for x in the original equation, $(x + 3)(2x - 1) = 0$. Then start over and substitute $\frac{1}{2}$ for x.

If $x = -3$, then

$$(x + 3)(2x - 1) = 0$$

$$(-3 + 3)[2(-3) - 1] = 0 \quad ?$$

$$0(-7) = 0. \qquad \text{True}$$

If $x = \frac{1}{2}$, then

$$(x + 3)(2x - 1) = 0$$

$$\left(\frac{1}{2} + 3\right)\left(2 \cdot \frac{1}{2} - 1\right) = 0 \quad ?$$

$$\frac{7}{2}(1 - 1) = 0 \quad ?$$

$$\frac{7}{2} \cdot 0 = 0. \qquad \text{True}$$

Both -3 and $\frac{1}{2}$ result in true equations, so the solution set is $\left\{-3, \frac{1}{2}\right\}$.

(b) $y(3y - 4) = 0$

$$y(3y - 4) = 0$$

$$y = 0 \quad \text{or} \quad 3y - 4 = 0 \qquad \text{Zero-factor property}$$

$$3y = 4$$

$$y = \frac{4}{3}$$

Check these solutions by substituting each one in the original equation. The solution set is $\left\{0, \frac{4}{3}\right\}$.

Now Try Exercises 3 and 5.

NOTE The word *or* as used in Example 1 means "one or the other or both."

In Example 1, each equation to be solved was given with the polynomial in factored form. If the polynomial in an equation is not already factored, first make sure that the equation is in standard form. Then factor.

VIDEO

EXAMPLE 2 Solving Quadratic Equations

Solve each equation.

(a) $x^2 - 5x = -6$

First, rewrite the equation in standard form by adding 6 to each side.

$$x^2 - 5x = -6$$

$$x^2 - 5x + 6 = 0 \qquad \text{Add 6.}$$

Now factor $x^2 - 5x + 6$. Find two numbers whose product is 6 and whose sum is -5. These two numbers are -2 and -3, so the equation becomes

$$(x - 2)(x - 3) = 0. \qquad \text{Factor.}$$
$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0 \qquad \text{Zero-factor property}$$
$$x = 2 \qquad\qquad x = 3 \qquad \text{Solve each equation.}$$

Check: If $x = 2$, then

$$x^2 - 5x = -6$$
$$2^2 - 5(2) = -6 \quad ?$$
$$4 - 10 = -6 \quad ?$$
$$-6 = -6. \quad \text{True}$$

If $x = 3$, then

$$x^2 - 5x = -6$$
$$3^2 - 5(3) = -6 \quad ?$$
$$9 - 15 = -6 \quad ?$$
$$-6 = -6. \quad \text{True}$$

Both solutions check, so the solution set is $\{2, 3\}$.

(b) $y^2 = y + 20$

Rewrite the equation in standard form.

$$y^2 = y + 20$$
$$y^2 - y - 20 = 0 \qquad \text{Subtract } y \text{ and } 20.$$
$$(y - 5)(y + 4) = 0 \qquad \text{Factor.}$$
$$y - 5 = 0 \quad \text{or} \quad y + 4 = 0 \qquad \text{Zero-factor property}$$
$$y = 5 \qquad\qquad y = -4 \qquad \text{Solve each equation.}$$

Check these solutions by substituting each one in the original equation. The solution set is $\{-4, 5\}$.

Now Try Exercise 21.

In summary, follow these steps to solve quadratic equations by factoring.

Solving a Quadratic Equation by Factoring

Step 1 **Write the equation in standard form,** that is, with all terms on one side of the equals sign in descending powers of the variable and 0 on the other side.

Step 2 **Factor** completely.

Step 3 **Use the zero-factor property** to set each factor with a variable equal to 0, and solve the resulting equations.

Step 4 **Check** each solution in the original equation.

NOTE Not all quadratic equations can be solved by factoring. A more general method for solving such equations is given in Chapter 9.

EXAMPLE 3 Solving a Quadratic Equation with a Common Factor

Solve $4p^2 + 40 = 26p$.

Subtract $26p$ from each side and write the equation in standard form to get

$$4p^2 - 26p + 40 = 0.$$

$$2(2p^2 - 13p + 20) = 0 \quad \text{Factor out 2.}$$
$$2p^2 - 13p + 20 = 0 \quad \text{Divide each side by 2.}$$
$$(2p - 5)(p - 4) = 0 \quad \text{Factor.}$$
$$2p - 5 = 0 \quad \text{or} \quad p - 4 = 0 \quad \text{Zero-factor property}$$
$$2p = 5 \qquad p = 4$$
$$p = \frac{5}{2}$$

Check that the solution set is $\left\{\frac{5}{2}, 4\right\}$ by substituting each solution in the original equation.

Now Try Exercise 31.

EXAMPLE 4 Solving Quadratic Equations

Solve each equation.

(a) $16m^2 - 25 = 0$

Factor the left side of the equation as the difference of squares.

$$(4m + 5)(4m - 5) = 0$$
$$4m + 5 = 0 \quad \text{or} \quad 4m - 5 = 0 \quad \text{Zero-factor property}$$
$$4m = -5 \qquad 4m = 5 \quad \text{Solve each equation.}$$
$$m = -\frac{5}{4} \qquad m = \frac{5}{4}$$

Check the two solutions, $-\frac{5}{4}$ and $\frac{5}{4}$, in the original equation. The solution set is $\left\{-\frac{5}{4}, \frac{5}{4}\right\}$.

(b) $y^2 = 2y$

First write the equation in standard form.

$$y^2 - 2y = 0 \quad \text{Standard form}$$
$$y(y - 2) = 0 \quad \text{Factor.}$$
$$y = 0 \quad \text{or} \quad y - 2 = 0 \quad \text{Zero-factor property}$$
$$y = 2 \quad \text{Solve.}$$

The solution set is $\{0, 2\}$.

(c) $k(2k + 1) = 3$

Write the equation in standard form.

$$k(2k + 1) = 3$$
$$2k^2 + k = 3 \quad \text{Distributive property}$$
$$2k^2 + k - 3 = 0 \quad \text{Subtract 3.}$$
$$(k - 1)(2k + 3) = 0 \quad \text{Factor.}$$
$$k - 1 = 0 \quad \text{or} \quad 2k + 3 = 0 \quad \text{Zero-factor property}$$
$$k = 1 \qquad 2k = -3$$
$$k = -\frac{3}{2}$$

The solution set is $\left\{1, -\frac{3}{2}\right\}$.

Now Try Exercises 37, 41, and 45.

CAUTION In Example 4(b) it is tempting to begin by dividing both sides of the equation $y^2 = 2y$ by y to get $y = 2$. Note that we do not get the other solution, 0, if we divide by a variable. (We may divide each side of an equation by a *nonzero* real number, however. For instance, in Example 3 we divided each side by 2.)

In Example 4(c) we could not use the zero-factor property to solve the equation $k(2k + 1) = 3$ in its given form because of the 3 on the right. Remember that the zero-factor property applies only to a product that equals 0.

OBJECTIVE 2 Solve other equations by factoring. We can also use the zero-factor property to solve equations that involve more than two factors with variables, as shown in Example 5. (These equations are *not* quadratic equations. Why not?)

EXAMPLE 5 Solving Equations with More Than Two Variable Factors

Solve each equation.

(a)

$$6z^3 - 6z = 0$$

$$6z(z^2 - 1) = 0 \qquad \text{Factor out } 6z.$$

$$6z(z + 1)(z - 1) = 0 \qquad \text{Factor } z^2 - 1.$$

By an extension of the zero-factor property, this product can equal 0 only if at least one of the factors is 0. Write and solve three equations, one for each factor with a variable.

$$6z = 0 \quad \text{or} \quad z + 1 = 0 \quad \text{or} \quad z - 1 = 0$$

$$z = 0 \qquad\qquad z = -1 \qquad\qquad z = 1$$

Check by substituting, in turn, 0, -1, and 1 in the original equation. The solution set is $\{-1, 0, 1\}$.

(b)

$$(3x - 1)(x^2 - 9x + 20) = 0$$

$$(3x - 1)(x - 5)(x - 4) = 0 \qquad \text{Factor } x^2 - 9x + 20.$$

$$3x - 1 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Zero-factor property}$$

$$x = \frac{1}{3} \qquad\qquad x = 5 \qquad\qquad x = 4$$

The solutions of the original equation are $\frac{1}{3}$, 4, and 5. Check each solution to verify that the solution set is $\left\{\frac{1}{3}, 4, 5\right\}$.

Now Try Exercises 51 and 55.

CAUTION In Example 5(b), it would be unproductive to begin by multiplying the two factors together. Keep in mind that the zero-factor property requires the *product* of two or more factors; this product must equal 0. Always consider first whether an equation is given in an appropriate form for the zero-factor property.

EXAMPLE 6 Solving an Equation Requiring Multiplication Before Factoring

Solve $(3x + 1)x = (x + 1)^2 + 5$.

The zero-factor property requires the *product* of two or more factors to equal 0. To write this equation in the required form, we must first multiply on both sides and collect terms on one side.

$$\begin{aligned}
(3x + 1)x &= (x + 1)^2 + 5 \\
3x^2 + x &= x^2 + 2x + 1 + 5 && \text{Multiply.} \\
3x^2 + x &= x^2 + 2x + 6 && \text{Combine like terms.} \\
2x^2 - x - 6 &= 0 && \text{Standard form} \\
(2x + 3)(x - 2) &= 0 && \text{Factor.} \\
2x + 3 = 0 \quad &\text{or} \quad x - 2 = 0 && \text{Zero-factor property} \\
x = -\frac{3}{2} \quad & \quad\quad\;\; x = 2
\end{aligned}$$

Check that the solution set is $\left\{-\frac{3}{2}, 2\right\}$.

Now Try Exercise 65.