9.1 Solving Quadratic Equations by the Square Root Property

OBJECTIVES

1. Solve equations of the form $x^2 = k$, where $k > 0$.
2. Solve equations of the form $(ax + b)^2 = k$, where $k > 0$.
3. Use formulas involving squared variables.

In Chapter 6 we solved quadratic equations by factoring. However, since not all quadratic equations can easily be solved by factoring, it is necessary to develop other methods. In this chapter we do just that.

Recall that a *quadratic equation* is an equation that can be written in standard form

$$ax^2 + bx + c = 0$$

for real numbers a, b, and c, with $a \neq 0$. As seen in Section 6.5, to solve the quadratic equation $x^2 + 4x + 3 = 0$ by the zero-factor property, we begin by factoring the left side and then setting each factor equal to 0.

$$x^2 + 4x + 3 = 0$$

$$(x + 3)(x + 1) = 0 \qquad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = -1 \qquad \text{Solve each equation.}$$

The solution set is $\{-3, -1\}$.

OBJECTIVE 1 Solve equations of the form $x^2 = k$, where $k > 0$. We can solve equations such as $x^2 = 9$ by factoring as follows.

$$x^2 = 9$$

$$x^2 - 9 = 0 \qquad \text{Subtract 9.}$$

$$(x + 3)(x - 3) = 0 \qquad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 3 = 0 \qquad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = 3$$

The solution set is $\{-3, 3\}$.

We might also have solved $x^2 = 9$ by noticing that x must be a number whose square is 9. Thus, $x = \sqrt{9} = 3$ or $x = -\sqrt{9} = -3$. This is generalized as the **square root property of equations.**

Square Root Property of Equations

If k is a positive number and if $x^2 = k$, then

$$\boldsymbol{x = \sqrt{k}} \quad \text{or} \quad \boldsymbol{x = -\sqrt{k},}$$

and the solution set is $\{-\sqrt{k}, \sqrt{k}\}$.

NOTE When we solve an equation, we want to find *all* values of the variable that satisfy the equation. Therefore, we want both the positive and negative square roots of k.

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EXAMPLE 1 Solving Quadratic Equations of the Form $x^2 = k$

Solve each equation. Write radicals in simplified form.

(a) $x^2 = 16$

By the square root property, since $x^2 = 16$,

$$x = \sqrt{16} = 4 \quad \text{or} \quad x = -\sqrt{16} = -4.$$

An abbreviation for "$x = 4$ or $x = -4$" is written $x = \pm 4$ (read "positive or negative 4"). Check each solution by substituting it for x in the original equation. The solution set is $\{-4, 4\}$.

(b) $z^2 = 5$

The solutions are $z = \sqrt{5}$ or $z = -\sqrt{5}$. The solution set may be written $\{\pm\sqrt{5}\}$.

(c)

$$\begin{aligned} 5m^2 - 40 &= 0 & & \\ 5m^2 &= 40 & & \text{Add 40.} \\ m^2 &= 8 & & \text{Divide by 5.} \\ m = \sqrt{8} \quad &\text{or} \quad m = -\sqrt{8} & & \text{Square root property} \\ m = 2\sqrt{2} \quad &\text{or} \quad m = -2\sqrt{2} & & \text{Simplify } \sqrt{8}. \end{aligned}$$

The solution set is $\{\pm 2\sqrt{2}\}$.

(d) $p^2 = -4$

Since -4 is a negative number and since the square of a real number cannot be negative, there is no real number solution for this equation. (The square root property cannot be used because of the requirement that k must be positive.) The solution set is $\emptyset$.

(e) $3x^2 + 5 = 11$

First solve the equation for x^2.

$$\begin{aligned} 3x^2 + 5 &= 11 & & \\ 3x^2 &= 6 & & \text{Subtract 5.} \\ x^2 &= 2 & & \text{Divide by 3.} \end{aligned}$$

Now use the square root property to get the solution set $\{\pm\sqrt{2}\}$.

Now Try Exercises 5, 7, 11, and 19.

OBJECTIVE 2 Solve equations of the form $(ax + b)^2 = k$, where $k > 0$. In each equation in Example 1, the exponent 2 appeared with a single variable as its base. We can extend the square root property to solve equations where the base is a binomial, as shown in the next example.

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EXAMPLE 2 Solving Quadratic Equations of the Form $(x + b)^2 = k$

Solve each equation.

(a) $(x - 3)^2 = 16$

Apply the square root property, using $x - 3$ as the base.

$$(x-3)^2 = 16$$

$$\begin{array}{rcl l}
x - 3 = \sqrt{16} & \text{or} & x - 3 = -\sqrt{16} & \\
x - 3 = 4 & \text{or} & x - 3 = -4 & \sqrt{16} = 4 \\
x = 7 & \text{or} & x = -1 & \text{Add 3.}
\end{array}$$

Check both answers in the original equation.

$$\begin{array}{rl|rl}
(x-3)^2 = 16 & & (x-3)^2 = 16 & \\
(7-3)^2 = 16 \;? & \text{Let } x = 7. & (-1-3)^2 = 16 \;? & \text{Let } x = -1. \\
4^2 = 16 \;? & & (-4)^2 = 16 \;? & \\
16 = 16 & \text{True} & 16 = 16 & \text{True}
\end{array}$$

The solution set is $\{7, -1\}$.

(b) $(x-1)^2 = 6$

By the square root property,

$$\begin{array}{rcl}
x - 1 = \sqrt{6} & \text{or} & x - 1 = -\sqrt{6} \\
x = 1 + \sqrt{6} & \text{or} & x = 1 - \sqrt{6}.
\end{array}$$

Check:

$$\left(1 + \sqrt{6} - 1\right)^2 = \left(\sqrt{6}\right)^2 = 6;$$

$$\left(1 - \sqrt{6} - 1\right)^2 = \left(-\sqrt{6}\right)^2 = 6.$$

The solution set is $\left\{1 + \sqrt{6}, 1 - \sqrt{6}\right\}$.

Now Try Exercises 23 and 27.

NOTE The solutions in Example 2(b) may be written in abbreviated form as

$$1 \pm \sqrt{6}.$$

If they are written this way, keep in mind that *two* solutions are indicated, one with the $+$ sign and the other with the $-$ sign.

EXAMPLE 3 Solving a Quadratic Equation of the Form $(ax + b)^2 = k$

Solve $(3r-2)^2 = 27$.

$$\begin{array}{rcl l}
3r - 2 = \sqrt{27} & \text{or} & 3r - 2 = -\sqrt{27} & \text{Square root property} \\
3r - 2 = 3\sqrt{3} & \text{or} & 3r - 2 = -3\sqrt{3} & \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3} \\
3r = 2 + 3\sqrt{3} & \text{or} & 3r = 2 - 3\sqrt{3} & \text{Add 2.} \\
r = \dfrac{2 + 3\sqrt{3}}{3} & \text{or} & r = \dfrac{2 - 3\sqrt{3}}{3} & \text{Divide by 3.}
\end{array}$$

The solution set is $\left\{\dfrac{2 \pm 3\sqrt{3}}{3}\right\}$.

Now Try Exercise 35.

CAUTION The solutions in Example 3 are fractions that cannot be simplified, since 3 is *not* a common factor in the numerator.

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EXAMPLE 4 Recognizing a Quadratic Equation with No Real Solutions

Solve $(x + 3)^2 = -9$.

Because the square root of -9 is not a real number, the solution set is $\emptyset$.

Now Try Exercise 25.

OBJECTIVE 3 Use formulas involving squared variables.

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EXAMPLE 5 Finding the Length of a Bass

The formula

$$w = \frac{L^2 g}{1200}$$

is used to approximate the weight of a bass, in pounds, given its length L and its girth g, where both are measured in inches. Approximate the length of a bass weighing 2.20 lb and having girth 10 in. (*Source: Sacramento Bee*, November 29, 2000.)

$$\begin{aligned} w &= \frac{L^2 g}{1200} && \text{Given formula} \\ 2.20 &= \frac{L^2 \cdot 10}{1200} && w = 2.20,\ g = 10 \\ 2640 &= 10L^2 && \text{Multiply by 1200.} \\ L^2 &= 264 && \text{Divide by 10.} \\ L &\approx 16.25 && \text{Use a calculator; } L > 0. \end{aligned}$$

The length of the bass is a little more than 16 in. (We discard the negative solution -16.25 since L represents length.)

Now Try Exercise 53.