9.3 Solving Quadratic Equations by the Quadratic Formula

OBJECTIVES

1. Identify the values of *a*, *b*, and *c* in a quadratic equation.
2. Use the quadratic formula to solve quadratic equations.
3. Solve quadratic equations with only one solution.
4. Solve quadratic equations with fractions.

We can solve any quadratic equation by completing the square, but the method is tedious. In this section we complete the square on the general quadratic equation $ax^2 + bx + c = 0$ to get the *quadratic formula*, a formula that gives the solutions for any quadratic equation. (Note that $a \neq 0$, or we would have a linear, not a quadratic, equation.)

OBJECTIVE 1 Identify the values of *a*, *b*, and *c* in a quadratic equation. The first step in solving a quadratic equation by this new method is to identify the values of a, b, and c in the standard form of the quadratic equation.

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EXAMPLE 1 Determining Values of *a*, *b*, and *c* in Quadratic Equations

Match the coefficients of each quadratic equation with the letters a, b, and c of the standard quadratic equation

$$ax^2 + bx + c = 0.$$

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(a) $\overset{a}{\downarrow}{2}x^2 + \overset{b}{\downarrow}{3}x - \overset{c}{\downarrow}{5} = 0$

In this example, $a = 2$, $b = 3$, and $c = -5$.

(b) $-x^2 + 2 = 6x$

First rewrite the equation with 0 on the right side to match the standard form $ax^2 + bx + c = 0$.

$$-x^2 + 2 = 6x$$

$$-x^2 - 6x + 2 = 0 \qquad \text{Subtract } 6x.$$

Here, $a = -1$, $b = -6$, and $c = 2$. (Notice that the coefficient of x^2 is understood to be -1.)

(c) $5x^2 - 12 = 0$

The x-term is missing, so write the equation as

$$5x^2 + 0x - 12 = 0.$$

Then $a = 5$, $b = 0$, and $c = -12$.

(d) $(2x - 7)(x + 4) = -23$

Write the equation in standard form.

$$(2x - 7)(x + 4) = -23$$
$$2x^2 + x - 28 = -23 \qquad \text{Use FOIL.}$$
$$2x^2 + x - 5 = 0 \qquad \text{Add 23.}$$

Now, identify the values: $a = 2$, $b = 1$, and $c = -5$.

Now Try Exercises 1, 7, and 9.

OBJECTIVE 2 Use the quadratic formula to solve quadratic equations. To develop the quadratic formula, we follow the steps given in the previous section for completing the square on $ax^2 + bx + c = 0$. For comparison, we also show the corresponding steps for solving $2x^2 + x - 5 = 0$ (from Example 1(d)).

Step 1 Make the coefficient of the squared term equal to 1.

$$2x^2 + x - 5 = 0 \qquad\qquad ax^2 + bx + c = 0$$
$$x^2 + \frac{1}{2}x - \frac{5}{2} = 0 \quad \text{Divide by 2.} \qquad\qquad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{Divide by } a.$$

Step 2 Get the variable terms alone on the left side.

$$x^2 + \frac{1}{2}x = \frac{5}{2} \quad \text{Add } \tfrac{5}{2}. \qquad\qquad x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Subtract } \tfrac{c}{a}.$$

Step 3 Add the square of half the coefficient of x to both sides, factor the left side, and combine terms on the right.

$$x^2 + \frac{1}{2}x + \frac{1}{16} = \frac{5}{2} + \frac{1}{16} \quad \text{Add } \tfrac{1}{16}. \qquad\qquad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \text{Add } \tfrac{b^2}{4a^2}.$$

$$\left(x + \frac{1}{4}\right)^2 = \frac{41}{16} \quad \text{Factor; add on right.} \qquad\qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Factor; add on right.}$$

Step 4 Use the square root property to complete the solution.

$$x + \frac{1}{4} = \pm\sqrt{\frac{41}{16}} \qquad\qquad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x + \frac{1}{4} = \pm\frac{\sqrt{41}}{4} \qquad\qquad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = -\frac{1}{4} \pm \frac{\sqrt{41}}{4} \qquad\qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{41}}{4} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The final result in the column on the right is called the **quadratic formula.** *It is a key result that should be memorized.* Notice that there are two values, one for the $+$ sign and one for the $-$ sign.

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

or, in compact form, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

CAUTION Notice that the fraction bar is under $-b$ as well as the radical. When using this formula, be sure to find the values of $-b \pm \sqrt{b^2 - 4ac}$ first, then divide those results by the value of $2a$.

EXAMPLE 2 Solving a Quadratic Equation by the Quadratic Formula

Use the quadratic formula to solve $2x^2 - 7x - 9 = 0$.

Match the coefficients of the variables with those of the standard quadratic equation

$$ax^2 + bx + c = 0.$$

Here, $a = 2$, $b = -7$, and $c = -9$. Substitute these numbers into the quadratic formula, and simplify the result.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-9)}}{2(2)} \qquad \text{Let } a = 2,\ b = -7,\ c = -9.$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{4}$$

$$x = \frac{7 \pm \sqrt{121}}{4}$$

$$x = \frac{7 \pm 11}{4} \qquad \sqrt{121} = 11$$

Find the two separate solutions by first using the plus sign, and then using the minus sign:

$$x = \frac{7 + 11}{4} = \frac{18}{4} = \frac{9}{2} \quad \text{or} \quad x = \frac{7 - 11}{4} = \frac{-4}{4} = -1.$$

Check by substituting each solution into the original equation. The solution set is $\left\{-1, \frac{9}{2}\right\}$.

Now Try Exercise 19.

EXAMPLE 3 Rewriting a Quadratic Equation Before Using the Quadratic Formula

Solve $x^2 = 2x + 1$.

Find a, b, and c by rewriting the equation $x^2 = 2x + 1$ in standard form (with 0 on one side). Add $-2x - 1$ to each side of the equation to get

$$x^2 - 2x - 1 = 0.$$

Then $a = 1$, $b = -2$, and $c = -1$. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \qquad \text{Let } a = 1,\ b = -2,\ c = -1.$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2 \pm 2\sqrt{2}}{2} \qquad \sqrt{8} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

Write the solutions in lowest terms by factoring $2 \pm 2\sqrt{2}$ as $2\left(1 \pm \sqrt{2}\right)$ to get

$$x = \frac{2\left(1 \pm \sqrt{2}\right)}{2} = 1 \pm \sqrt{2}. \qquad \text{Factor, then divide out.}$$

The solution set is $\left\{1 \pm \sqrt{2}\right\}$.

Now Try Exercise 21.

OBJECTIVE 3 Solve quadratic equations with only one solution. In the quadratic formula, the quantity under the radical, $b^2 - 4ac$, is called the **discriminant.** When the discriminant equals 0, the equation has just one rational number solution. In this case, the trinomial $ax^2 + bx + c$ is a perfect square.

EXAMPLE 4 Solving a Quadratic Equation with Only One Solution

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Solve $4x^2 + 25 = 20x$.

Write the equation as

$$4x^2 - 20x + 25 = 0. \qquad \text{Subtract } 20x.$$

Here, $a = 4$, $b = -20$, and $c = 25$. By the quadratic formula,

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 400}}{8} = \frac{20 \pm 0}{8} = \frac{5}{2}.$$

In this case, $b^2 - 4ac = 0$, and the trinomial $4x^2 - 20x + 25$ is a perfect square. There is just one solution in the solution set $\left\{\frac{5}{2}\right\}$.

Now Try Exercise 17.

NOTE The single solution of the equation in Example 4 is a rational number. If all solutions of a quadratic equation are rational, the equation can be solved by factoring as well.

OBJECTIVE 4 Solve quadratic equations with fractions. It is usually easier to clear quadratic equations of fractions before solving them, as shown in the next example.

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EXAMPLE 5 Solving a Quadratic Equation with Fractions

Solve $\frac{1}{10}t^2 = \frac{2}{5}t - \frac{1}{2}$.

Eliminate the denominators by multiplying both sides of the equation by the least common denominator, 10.

$$10\left(\frac{1}{10}t^2\right) = 10\left(\frac{2}{5}t - \frac{1}{2}\right)$$

$$t^2 = 4t - 5 \qquad \text{Distributive property}$$

$$t^2 - 4t + 5 = 0 \qquad \text{Standard form}$$

From this form identify $a = 1$, $b = -4$, and $c = 5$. Use the quadratic formula to complete the solution.

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \qquad \text{Substitute into the formula.}$$

$$t = \frac{4 \pm \sqrt{16 - 20}}{2} \qquad \text{Perform the operations.}$$

$$t = \frac{4 \pm \sqrt{-4}}{2}$$

The discriminant -4 is less than 0. Because $\sqrt{-4}$ does not represent a real number, the solution set is $\emptyset$.

Now Try Exercise 49.