6.3 More on Factoring Trinomials

OBJECTIVES

1. Factor trinomials by grouping when the coefficient of the squared term is not 1.
2. Factor trinomials using FOIL.

Trinomials like $2x^2 + 7x + 6$, in which the coefficient of the squared term is *not* 1, are factored with extensions of the methods from the previous sections. One such method uses factoring by grouping from Section 6.1.

OBJECTIVE 1 Factor trinomials by grouping when the coefficient of the squared term is not 1. Recall that a trinomial such as $m^2 + 3m + 2$ is factored by finding two numbers whose product is 2 and whose sum is 3. To factor $2x^2 + 7x + 6$, we look for two integers whose product is $2 \cdot 6 = 12$ and whose sum is 7.

Sum is 7.

$$2x^2 + 7x + 6$$

Product is $2 \cdot 6 = 12$.

By considering pairs of positive integers whose product is 12, we find the necessary integers to be 3 and 4. We use these integers to write the middle term, $7x$, as $7x = 3x + 4x$. The trinomial $2x^2 + 7x + 6$ becomes

$$
\begin{aligned}
2x^2 + 7x + 6 &= 2x^2 + \underbrace{3x + 4x}_{7x} + 6. \\
&= (2x^2 + 3x) + (4x + 6) && \text{Group terms.} \\
&= x(2x + 3) + 2(2x + 3) && \text{Factor each group.}
\end{aligned}
$$

Must be the same factor

$$2x^2 + 7x + 6 = (2x + 3)(x + 2) \qquad \text{Factor out } 2x + 3.$$

Check: $(2x + 3)(x + 2) = 2x^2 + 7x + 6$

In this example, we could have written $7x$ as $4x + 3x$. Factoring by grouping this way would give the same answer.

EXAMPLE 1 Factoring Trinomials by Grouping

Factor each trinomial.

(a) $6r^2 + r - 1$

We must find two integers with a product of $6(-1) = -6$ and a sum of 1.

Sum is 1.

$$6r^2 + r - 1 = 6r^2 + 1r - 1$$

Product is $6(-1) = -6$.

The integers are -2 and 3. We write the middle term, r, as $-2r + 3r$.

$$
\begin{aligned}
6r^2 + r - 1 &= 6r^2 - 2r + 3r - 1 && r = -2r + 3r \\
&= (6r^2 - 2r) + (3r - 1) && \text{Group terms.} \\
&= 2r(3r - 1) + 1(3r - 1) && \text{The binomials must be the same.} \\
&= (3r - 1)(2r + 1) && \text{Factor out } 3r - 1.
\end{aligned}
$$

Check: $(3r - 1)(2r + 1) = 6r^2 + r - 1$

(b) $12z^2 - 5z - 2$

Look for two integers whose product is $12(-2) = -24$ and whose sum is -5. The required integers are 3 and -8, so

$$
\begin{aligned}
12z^2 - 5z - 2 &= 12z^2 + 3z - 8z - 2 && -5z = 3z - 8z \\
&= (12z^2 + 3z) + (-8z - 2) && \text{Group terms.} \\
&= 3z(4z + 1) - 2(4z + 1) && \text{Factor each group; be careful with signs.} \\
&= (4z + 1)(3z - 2). && \text{Factor out } 4z + 1.
\end{aligned}
$$

Check: $(4z + 1)(3z - 2) = 12z^2 - 5z - 2$

(c) $10m^2 + mn - 3n^2$

Two integers whose product is $10(-3) = -30$ and whose sum is 1 are -5 and 6. Rewrite the trinomial with four terms.

$$\begin{aligned} 10m^2 + mn - 3n^2 &= 10m^2 - 5mn + 6mn - 3n^2 && mn = -5mn + 6mn \\ &= 5m(2m - n) + 3n(2m - n) && \text{Group terms; factor each group.} \\ &= (2m - n)(5m + 3n) && \text{Factor out } 2m - n. \end{aligned}$$

Check by multiplying.

Now Try Exercises 21, 27, and 47.

EXAMPLE 2 Factoring a Trinomial with a Common Factor by Grouping

Factor $28x^5 - 58x^4 - 30x^3$.

First factor out the greatest common factor, $2x^3$.

$$28x^5 - 58x^4 - 30x^3 = 2x^3(14x^2 - 29x - 15)$$

To factor $14x^2 - 29x - 15$, find two integers whose product is $14(-15) = -210$ and whose sum is -29. Factoring 210 into prime factors gives

$$210 = 2 \cdot 3 \cdot 5 \cdot 7.$$

Combine these prime factors in pairs in different ways, using one positive and one negative (to get -210). The factors 6 and -35 have the correct sum. Now rewrite the given trinomial and factor it.

$$\begin{aligned} 28x^5 - 58x^4 - 30x^3 &= 2x^3(14x^2 + 6x - 35x - 15) \\ &= 2x^3[(14x^2 + 6x) + (-35x - 15)] \\ &= 2x^3[2x(7x + 3) - 5(7x + 3)] \\ &= 2x^3[(7x + 3)(2x - 5)] \\ &= 2x^3(7x + 3)(2x - 5) \end{aligned}$$

Check by multiplying.

Now Try Exercise 43.

CAUTION Remember to include the common factor in the final result.

OBJECTIVE 2 Factor trinomials using FOIL. In the rest of this section we show an alternative method of factoring trinomials in which the coefficient of the squared term is not 1. This method generalizes the factoring procedure explained in Section 6.2.

To factor $2x^2 + 7x + 6$ (the trinomial factored at the beginning of this section) by the method in Section 6.2, use FOIL backwards. We want to write $2x^2 + 7x + 6$ as the product of two binomials.

$$2x^2 + 7x + 6 = (\qquad)(\qquad)$$

The product of the two first terms of the binomials is $2x^2$. The possible factors of $2x^2$ are $2x$ and x or $-2x$ and $-x$. Since all terms of the trinomial are positive, we consider only positive factors. Thus, we have

$$2x^2 + 7x + 6 = (2x \qquad)(x \qquad).$$

The product of the two last terms, 6, can be factored as $1 \cdot 6$, $6 \cdot 1$, $2 \cdot 3$, or $3 \cdot 2$. Try each pair to find the pair that gives the correct middle term, $7x$.

$(2x + 1)(x + 6)$ Incorrect

x
$12x$
$13x$ Add.

$(2x + 6)(x + 1)$ Incorrect

$6x$
$2x$
$8x$ Add.

Since $2x + 6 = 2(x + 3)$, the binomial $2x + 6$ has a common factor of 2, while $2x^2 + 7x + 6$ has no common factor other than 1. The product $(2x + 6)(x + 1)$ cannot be correct.

NOTE If the original polynomial has no common factor, then none of its binomial factors will either.

Now try the numbers 2 and 3 as factors of 6. Because of the common factor of 2 in $2x + 2$, $(2x + 2)(x + 3)$ will not work, so we try $(2x + 3)(x + 2)$.

$(2x + 3)(x + 2) = 2x^2 + 7x + 6$ Correct

$3x$
$4x$
$7x$ Add.

Thus, $2x^2 + 7x + 6$ factors as

$$2x^2 + 7x + 6 = (2x + 3)(x + 2).$$

Check by multiplying $2x + 3$ and $x + 2$.

VIDEO

YOU TRY IT

EXAMPLE 3 Factoring a Trinomial with All Positive Terms Using FOIL

Factor $8p^2 + 14p + 5$.

The number 8 has several possible pairs of factors, but 5 has only 1 and 5 or -1 and -5. For this reason, it is easier to begin by considering the factors of 5. Ignore the negative factors since all coefficients in the trinomial are positive. If $8p^2 + 14p + 5$ can be factored, the factors will have the form

$$(\quad + 5)(\quad + 1).$$

The possible pairs of factors of $8p^2$ are $8p$ and p, or $4p$ and $2p$. Try various combinations, checking in each case to see if the middle term is $14p$.

$(8p + 5)(p + 1)$ Incorrect

$5p$
$8p$
$13p$ Add.

$(p + 5)(8p + 1)$ Incorrect

$40p$
p
$41p$ Add.

$(4p + 5)(2p + 1)$ Correct

$10p$
$4p$
$14p$ Add.

Since $14p$ is the correct middle term,

$$8p^2 + 14p + 5 = (4p + 5)(2p + 1).$$

Check: $(4p + 5)(2p + 1) = 8p^2 + 14p + 5$

Now Try Exercise 23.

EXAMPLE 4 Factoring a Trinomial with a Negative Middle Term Using FOIL

Factor $6x^2 - 11x + 3$.

Since 3 has only 1 and 3 or -1 and -3 as factors, it is better here to begin by factoring 3. The last term of the trinomial $6x^2 - 11x + 3$ is positive and the middle term has a negative coefficient, so we consider only negative factors. We need two negative factors because the *product* of two negative factors is positive and their *sum* is negative, as required. Use -3 and -1 as factors of 3:

$$(\quad - 3)(\quad - 1).$$

The factors of $6x^2$ may be either $6x$ and x, or $2x$ and $3x$. We try $2x$ and $3x$.

$$(2x - 3)(3x - 1) \qquad \text{Correct}$$

$$\begin{array}{r} -9x \\ -2x \\ \hline -11x \end{array} \qquad \text{Add.}$$

These factors give the correct middle term, so

$$6x^2 - 11x + 3 = (2x - 3)(3x - 1).$$

Check by multiplying.

Now Try Exercise 29.

EXAMPLE 5 Factoring a Trinomial with a Negative Last Term Using FOIL

Factor $8x^2 + 6x - 9$.

The integer 8 has several possible pairs of factors, as does -9. Since the last term is negative, one positive factor and one negative factor of -9 are needed. Since the coefficient of the middle term is small, it is wise to avoid large factors such as 8 or 9. We try $4x$ and $2x$ as factors of $8x^2$, and 3 and -3 as factors of -9, and check the middle term.

$$(4x + 3)(2x - 3) \qquad \text{Incorrect}$$

$$\begin{array}{r} 6x \\ -12x \\ \hline -6x \end{array} \qquad \text{Add.}$$

Now, we try interchanging 3 and -3, since only the sign of the middle term is incorrect.

$$(4x - 3)(2x + 3) \qquad \text{Correct}$$

$$\begin{array}{r} -6x \\ 12x \\ \hline 6x \end{array} \qquad \text{Add.}$$

This combination produces the correct middle term, so

$$8x^2 + 6x - 9 = (4x - 3)(2x + 3).$$

Now Try Exercise 33.

EXAMPLE 6 Factoring a Trinomial with Two Variables

Factor $12a^2 - ab - 20b^2$.

There are several pairs of factors of $12a^2$, including $12a$ and a, $6a$ and $2a$, and $3a$ and $4a$, just as there are many pairs of factors of $-20b^2$, including $20b$ and $-b$,

$-20b$ and b, $10b$ and $-2b$, $-10b$ and $2b$, $4b$ and $-5b$, and $-4b$ and $5b$. Once again, since the desired middle term is small, avoid the larger factors. Try the factors $6a$ and $2a$, and $4b$ and $-5b$.

$$(6a + 4b)(2a - 5b)$$

This cannot be correct, as mentioned before, since $6a + 4b$ has a common factor while the given trinomial has none. Try $3a$ and $4a$ with $4b$ and $-5b$.

$$(3a + 4b)(4a - 5b) = 12a^2 + ab - 20b^2 \qquad \text{Incorrect}$$

Here the middle term has the wrong sign, so we interchange the signs in the factors.

$$(3a - 4b)(4a + 5b) = 12a^2 - ab - 20b^2 \qquad \text{Correct}$$

Now Try Exercise 41.

EXAMPLE 7 Factoring Trinomials with Common Factors

EXPLORE

Factor each trinomial.

(a) $15y^3 + 55y^2 + 30y$

First factor out the greatest common factor, $5y$.

$$15y^3 + 55y^2 + 30y = 5y(3y^2 + 11y + 6)$$

Now factor $3y^2 + 11y + 6$. Try $3y$ and y as factors of $3y^2$, and 2 and 3 as factors of 6.

$$(3y + 2)(y + 3) = 3y^2 + 11y + 6 \qquad \text{Correct}$$

The complete factored form of $15y^3 + 55y^2 + 30y$ is

$$15y^3 + 55y^2 + 30y = 5y(3y + 2)(y + 3).$$

Check by multiplying.

(b) $-24a^3 - 42a^2 + 45a$

The common factor could be $3a$ or $-3a$. If we factor out $-3a$, the first term of the trinomial will be positive, which makes it easier to factor.

$$\begin{aligned} -24a^3 - 42a^2 + 45a &= -3a(8a^2 + 14a - 15) && \text{Factor out } -3a. \\ &= -3a(4a - 3)(2a + 5) && \text{Use FOIL.} \end{aligned}$$

Check by multiplying.

Now Try Exercise 45.

CAUTION This caution bears repeating: Remember to include the common factor in the final factored form.

The two methods for factoring trinomials are summarized here.

Factoring Trinomials of the Form $ax^2 + bx + c$

Write the middle term of the trinomial as the sum of two terms to get a polynomial with four terms. Use factoring by grouping as shown in Section 6.1.

Factor the trinomial using FOIL backwards as explained in Section 6.2 and this section.