QUIZ #3 @ 85 points

Write neatly. Show all work. **Write all responses on separate paper.** **Clearly label the exercises.**

1. If $\int_1^x f(t)\,dt = x^2 - 3x + 5$, find $f(x)$.

2. Find $\frac{dy}{dx}$ if $y = \int_{1+3x^2}^{4} \frac{1}{2+e^t}\,dt$.

3. Suppose that f is a differentiable function shown in the accompanying graph and that the **position at time t (sec) of a particle moving along a coordinate axis is**

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. **Give reasons for your answers.**

a) What does the function shown in the graph represent in the context of the problem?

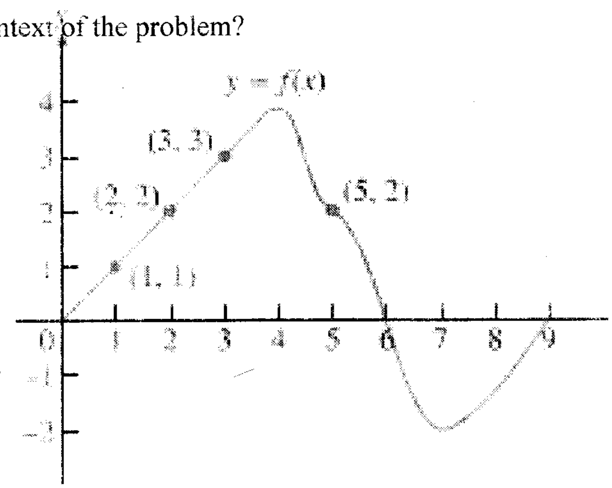
b) What is the particle's velocity at $t = 5$?

c) Is the acceleration at time $t = 5$ positive or negative?

d) What is the particle's position at time $t = 3$?

e) Approximately when is the acceleration zero?

f) On which side of the origin does the particle lie at time $t = 9$?



4) Evaluate the integral and give its geometrical interpretation. Illustrate with a sketch $\int_{-1}^{2} x^3\,dx$

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Find the following. Do not just write an answer. **Show work to justify your answer.**

5) $\int_0^3 \frac{1}{1+4x}\,dx$

6) $\int \frac{\sin 2\theta}{\sin\theta}\,d\theta$

7) $\int_{-4}^{-2} \frac{1}{x}\,dx$

8) $\int \sin^2 x\,dx$

9) $\int \sec^2 y\,dy$

10) $\int_1^3 x\ln x\,dx$

11) $\int \frac{\sin 2\theta}{1+\cos^2\theta}\,d\theta$

12) $\int_0^1 \sqrt[3]{1+7x}\,dx$

13) $\int \tan x\,dx$

14) $\int \sec y\,dy$

15) $\int_1^5 \frac{\ln r}{r^2}\,dr$

16) $\int e^{\sqrt{x}}\,dx$

17) $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

18) $\int_0^{\frac{1}{\sqrt{2}}} 2x\sin^{-1}\left(x^2\right)dx$

19) $\int \cos\sqrt{x}\,dx$

Extra Credit

1) Find $\int_2^4 x^{2x}\left(1+\ln x\right)dx$

2) Find $\int \cos^4 x\,dx$.

M180 - Fall 2017

Quiz 3 - SOLUTIONS

① $\int_1^x f(t)\,dt = x^2 - 3x + 5$

then

$$\frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}\left(x^2 - 3x + 5\right)$$

$$\boxed{f(x) = 2x - 3}$$

② $y = \int_{1+3x^2}^{4} \frac{1}{2+e^t}\,dt$

$$y = -\int_4^{1+3x^2} \frac{1}{2+e^t}\,dt$$

let $1+3x^2 = u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}\left(-\int_4^u \frac{1}{2+e^t}\,dt\right)\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2+e^u}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{2+e^{1+3x^2}}\cdot 6x$$

$$\boxed{\frac{dy}{dx} = \frac{-6x}{2+e^{1+3x^2}}}$$

③ (a) $s(t) = \int_0^t f(x)\,dx$

position at time t

$$\frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t)$$

so the graph represent the velocity function

$$\boxed{f(t) = v(t) = \frac{ds}{dt}}$$

(b) $\boxed{v(5) = f(5) = 2}$

(c) $a(t) = \frac{dv}{dt}$, so $a(5) < 0$

as the slope of the tangent to the curve is negative

(d) $s(3) = \int_0^3 f(x)\,dx = \text{Area}(\Delta)$

(as $f(x) \geq 0$ on $[0,3]$)

$$s(3) = \frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}$$

$\boxed{s(3) = 4.5 \text{ units}}$

(e) $a(4) = 0$
$a(7) = 0$ as the tangent to the graph is horizontal at about $t = 4$ seconds, $t = 7$ sec.

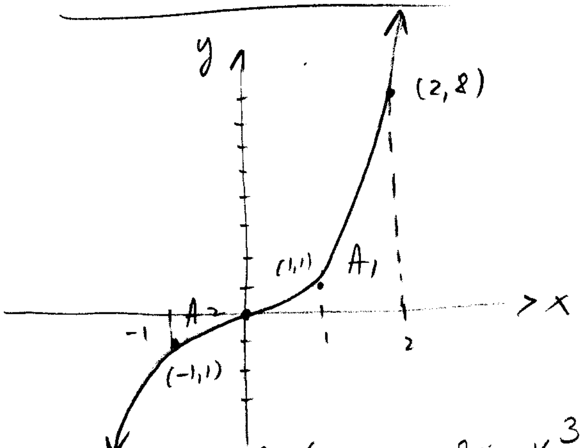
(f) $s(9) = \int_0^9 f(x)\,dx = A_{above} - A_{below}$

so $s(9) > 0$, so $\boxed{\text{On the right}}$

(4) $I = \int_{-1}^{2} x^3\,dx$

$I = \frac{x^4}{4}\Big]_{-1}^{2} = \frac{1}{4}\left(2^4 - (-1)^4\right)$

$$\boxed{\int_{-1}^{2} x^3\,dx = \frac{15}{4}}$$



A_1 = area between $y = x^3$ and the x-axis when $x \in [0,2]$

A_2 = area between $y = x^3$ and the x-axis when $x \in [-1,0]$

$$\boxed{\int_{-1}^{2} x^3\,dx = A_1 - A_2 = \frac{15}{4}}$$

(5) $\int_0^3 \frac{1}{1+4x}\,dx = I$

Substitution method

let $1 + 4x = u$

$4\,dx = du \Rightarrow dx = \frac{1}{4}\,du$

when $x = 0,\ u = 1$

$x = 3,\ u = 13$

$I = \frac{1}{4}\int_1^{13} \frac{1}{u}\,du = \frac{1}{4}\ln|u|\Big]_1^{13}$

$I = \frac{1}{4}(\ln 13 - \ln 1) = \boxed{\frac{\ln 13}{4}}$

(6) $\int \frac{\sin 2\theta}{\sin\theta}\,d\theta =$

$= \int \frac{2\sin\theta\cos\theta}{\sin\theta}\,d\theta$

$= 2\int \cos\theta\,d\theta = \boxed{2\sin\theta + C}$

where $C \in \mathbb{R}$

(7) $\int_{-4}^{-2} \frac{1}{x}\,dx = \ln|x|\Big]_{-4}^{-2}$

$= \ln|-2| - \ln|-4|$

$= \ln 2 - \ln 4 = \ln 2 - 2\ln 2$

$= \boxed{-\ln 2}$

(8) $\int \sin^2 x\,dx = I$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$I = \int \frac{1-\cos 2x}{2}\,dx$

$I = \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 2x\,dx$

$I = \frac{1}{2}x + C_1 - \frac{1}{2}\left(\frac{\sin 2x}{2} + C_2\right)$

$$\boxed{I = \frac{x}{2} - \frac{\sin 2x}{4} + C,\ C \in \mathbb{R}}$$

(9) $\int \sec^2 y \, dy = \boxed{\tan y + c}$

$c \in \mathbb{R}$

(10) $I = \int_1^3 x \ln x \, dx$

integration by parts

$\left[\begin{array}{ll} \text{let } f = \ln x & g' = x \\ \text{then } f' = \frac{1}{x} & g = \frac{x^2}{2} \end{array}\right.$

$I = \frac{x^2}{2} \ln x \Big]_1^3 - \frac{1}{2} \int_1^3 \frac{1}{x} \cdot x^2 dx$

$I = \frac{x^2}{2} \ln x \Big]_1^3 - \frac{1}{2} \cdot \frac{x^2}{2} \Big]_1^3$

$I = \frac{9}{2} \ln 3 - \frac{1}{2} \ln 1 - \left(\frac{9}{4} - \frac{1}{4}\right)$

$\boxed{I = \frac{9}{2} \ln 3 - 2}$

(11) $I = \int \frac{\sin 2\theta}{1 + \cos^2\theta} d\theta$

$I = \int \frac{2 \sin\theta \cos\theta}{1 + \cos^2\theta} d\theta$

Substitution method

$\left\{\begin{array}{l} \text{let } 1 + \cos^2\theta = u > 0 \\ -2\cos\theta \sin\theta \, d\theta = du \\ 2\sin\theta \cos\theta \, d\theta = -du \end{array}\right.$

$I = \int \frac{-du}{u} = -\ln|u| + c$

$\boxed{I = -\ln(1 + \cos^2\theta) + c}$

(12) $I = \int_0^1 \sqrt[3]{1 + 7x} \, dx$

Substitution method

$\left\{\begin{array}{l} \text{let } 1 + 7x = u \\ \quad 7dx = du \Rightarrow dx = \frac{1}{7} du \\ \text{when } x = 0, u = 1 \\ \quad x = 1, u = 8 \end{array}\right.$

$I = \frac{1}{7} \int_1^8 u^{\frac{1}{3}} du = \frac{1}{7} \cdot \frac{u^{\frac{1}{3}+1}}{\frac{4}{3}} \Big]_1^8$

$I = \frac{3}{28} u^{\frac{4}{3}} \Big]_1^8 = \frac{3}{28} \left(8^{\frac{4}{3}} - 1^{\frac{4}{3}}\right)$

note that $8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$

so $\boxed{I = \frac{45}{28}}$

(13) $I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

Substitution method

$\left\{\begin{array}{l} \text{let } \cos x = u \\ \quad -\sin x \, dx = du \\ \quad \sin x \, dx = -du \end{array}\right.$

$I = \int \frac{-du}{u} = -\ln|u| + c$

$\boxed{I = -\ln|\cos x| + c}$

(14) $I = \int \sec y \, dy$

$$I = \int \frac{\sec y(\sec y + \tan y)}{\sec y + \tan y} dy$$

$$I = \int \frac{\sec^2 y + \sec y \tan y}{\sec y + \tan y} dy$$

substitution method

$$\begin{cases} \text{let } \sec y + \tan y = u \\ (\sec y \tan y + \sec^2 y) dy = du \end{cases}$$

$$I = \int \frac{du}{u} = \ln|u| + C$$

$$\boxed{I = \ln|\sec y + \tan y| + C}$$

(15) $I = \int_1^5 \frac{\ln r}{r^2} dr$

integration by parts:

$$\left[\begin{array}{ll} \text{let } f = \ln r & g' = r^{-2} \\ f' = \frac{1}{r} & g = -r^{-1} \end{array}\right.$$

$$I = -r^{-1}\ln r\Big]_1^5 - \int_1^5 -\frac{1}{r} r^{-1} dr$$

$$I = \frac{-1}{r}\ln r\Big]_1^5 + \int_1^5 r^{-2} dr$$

$$I = \frac{-1}{r}\ln r\Big]_1^5 + \frac{r^{-1}}{-1}\Big]_1^5$$

$$I = \frac{-1}{5}\ln 5 - \left(\frac{-1}{1}\ln 1\right) - (5^{-1} - 1^{-1})$$

$$I = \frac{-1}{5}\ln 5 + 0 - \left(\frac{1}{5} - 1\right)$$

$$I = \frac{-1}{5}\ln 5 + \frac{4}{5}$$

$$\boxed{I = \frac{-1}{5}\ln 5 + \frac{4}{5}}$$

(16) $\int e^{\sqrt{x}} dx = I$

substitution method

$$\begin{cases} \text{let } \sqrt{x} = u \\ \frac{1}{2\sqrt{x}} dx = du \\ dx = 2\sqrt{x}\, du \\ dx = 2u\, du \end{cases}$$

$$I = \int 2u e^u du = 2\int u e^u du$$

integration by Parts

$$\text{let}\left[\begin{array}{ll} f = u & g' = e^u \\ f' = 1 & g = e^u \end{array}\right.$$

$$I = 2\left(e^u u - \int e^u du\right)$$

$$I = 2e^u u - 2e^u + C$$

$$\boxed{I = 2e^{\sqrt{x}}\sqrt{x} - 2e^{\sqrt{x}} + C}$$

$i = u\ln u - \int du$

$i = u\ln u - u + c$

$i = e^{\sqrt{x}}\ln e^{\sqrt{x}} - e^{\sqrt{x}} + c$

$$\boxed{i = \sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + C}$$

(17) $i = \int_0^{\pi} \sqrt{1-\cos 2x}\, dx$

$$\begin{cases} \cos 2x = 2\cos^2 x - 1 \\ \cos 2x = 1 - 2\sin^2 x \Rightarrow \\ 1 - \cos 2x = 2\sin^2 x \end{cases}$$

$i = \int_0^{\pi} \sqrt{2\sin^2 x}\, dx =$

$= \sqrt{2}\int_0^{\pi} |\sin x|\, dx$

but $\sin x \geq 0$, $\forall x \in [0,\pi]$

$i = \sqrt{2}\int_0^{\pi} \sin x\, dx$

$i = \sqrt{2}\left(-\cos x\right)\Big]_0^{\pi}$

$i = -\sqrt{2}(\cos\pi - \cos 0)$

$$\boxed{i = 2\sqrt{2}}$$

(18) $i = \int_0^{\frac{1}{\sqrt{2}}} 2x\sin^{-1}(x^2)\, dx$

$$\begin{cases} \text{let } x^2 = u \\ 2x\,dx = du \\ \text{when } x = 0,\ u = 0 \\ x = \frac{1}{\sqrt{2}},\ u = \frac{1}{2} \end{cases}$$

$i = \int_0^{\frac{1}{2}} \sin^{-1}u\, du$

integration by parts

$$\begin{bmatrix} f = \sin^{-1}u & \longrightarrow & g' = 1 \\ f' = \frac{1}{\sqrt{1-u^2}} & \longleftarrow & g = u \end{bmatrix}$$

$i = u\sin^{-1}u\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{u}{\sqrt{1-u^2}}\, du$

$$\begin{cases} \text{let } \sqrt{1-u^2} = t \\ \frac{-2u}{2\sqrt{1-u^2}}\, du = dt \\ \frac{-u}{\sqrt{1-u^2}}\, du = dt \\ \text{when } u = 0,\ t = 1 \\ u = \frac{1}{2},\ t = \frac{\sqrt{3}}{2} \end{cases}$$

$i = u\sin^{-1}u\Big]_0^{1/2} + \int_1^{\frac{\sqrt{3}}{2}} dt$

$i = u\sin^{-1}u\Big]_0^{1/2} + t\Big]_1^{\frac{\sqrt{3}}{2}}$

$i = \frac{1}{2}\sin^{-1}\frac{1}{2} - 0 + \frac{\sqrt{3}}{2} - 1$

$i = \frac{1}{2}\,\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 = \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1}$

(19) $i = \int \cos\sqrt{x}\, dx$

let $\sqrt{x} = u$

$\frac{1}{2\sqrt{x}} dx = du \Rightarrow$

$dx = 2\sqrt{x}\, du$

$dx = 2u\, du$

$i = \int 2u \cos u\, du$

int. by Parts:

let $f = u \quad g' = \cos u$

$f' = 1 \quad g = \sin u$

$i = 2\left(u \sin u - \int \sin u\, du\right)$

$i = 2u \sin u - 2(-\cos u) + c$

$i = 2u \sin u + 2\cos u + c$

$\boxed{i = 2\sqrt{x} \sin\sqrt{x} + 2\cos\sqrt{x} + c}$

EXTRA CREDIT

(1) $i = \int_2^4 x^{2x}(1 + \ln x)\, dx$

let $u = x^{2x}$

then $\ln u = \ln(x^{2x})$

$\ln u = 2x \ln x$

$\frac{d}{dx}(\ln u) = \frac{d}{dx}(2x \ln x)$

$\frac{1}{u}\frac{du}{dx} = 2\ln x + 2x \cdot \frac{1}{x}$

$\frac{1}{u}\frac{du}{dx} = 2(\ln x + 1)$

$\frac{1}{2} du = x^{2x}(\ln x + 1)\, dx$

when $x = 2$, $u = 2^4 = 16$

$x = 4$, $u = 4^8$

$i = \int_2^4 x^{2x}(1 + \ln x)\, dx$

$= \int_{16}^{4^8} \frac{1}{2} du = \frac{1}{2} u \Big]_{16}^{4^8} = \frac{1}{2}(4^8 - 16)$

$\boxed{i = 32{,}760}$

(2) $i = \int \cos^4 dx$

Method I

$i = \int (\cos^2 x)^2 dx$

know $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$i = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx$

$i = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$

know $\cos 4x = 2\cos^2 2x - 1$

$\cos^2 2x = \frac{1 + \cos 4x}{2}$

$i = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\, dx +$

$+ \frac{1}{4}\int \frac{1 + \cos 4x}{2} dx$

$i = \frac{1}{4}x + c_1 + \frac{1}{4}\sin 2x + c_2 +$

$+ \frac{1}{8}\left(x + \frac{\sin 4x}{4} + c_3\right)$

$i = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$