TEST #2 @ 145 points

Write neatly. Show all work. **Write all proofs on separate green paper. Label each exercise.**

1) Complete the following definitions, properties, theorems, or formulas:

a) The Max – Min Inequality for Definite Integrals:

If $m \le f(x) \le M$ for $x \in [a,b]$, then ____

b) Write the definition of the definite integral using Riemann sums and support your answer with a graph.

c) What is a critical number for a function defined on an interval I ?

d) $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt =$ ________________

e) What is an inflection point?

f) $\dfrac{d}{dx}\left(\log_a x\right) =$ __________________

g) $\displaystyle\int x^n\,dx =$ ______________________

h) $\displaystyle\int \frac{1}{x}\,dx =$ ______________________

2) Find the critical numbers for the following function:

$$f(x) = \sqrt[3]{x^2 - x}$$

3) Find the absolute minimum and maximum values for the following function on the given interval:

$$f(x) = x - 2\sin x,\ x \in [0, 3\pi]$$

4) Let $f(x)=\dfrac{x}{(1+x)^2}$. Do the following:

a) Graph the following function. Show: end behavior, behavior near vertical asymptotes (if any), intercepts, first and second derivative. Show all work and organize the information in a table. Label all points used.

b) What are the maximum and minimum values of the function?

c) What are the inflection points?

d) On what interval(s) is the function increasing? Decreasing?

e) On what interval(s) is the function concave up? Down?

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5) Find the following limits:

a) $\lim\limits_{x\to\infty}\left(\ln 2x-\ln(x+1)\right)$

b) $\lim\limits_{x\to 0}(1-2x)^{\frac{1}{x}}$

c) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x$

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6) Find the following:

a) $\int \ln x dx$

b) $\int_{-1}^{3}\left(4+3x^3\right)dx$

c) $\int_{1}^{2}\left(\dfrac{1}{x}-e^{-x}\right)dx$

d) $\int\left(2^{t+1}-\sin t\right)dt$

e) $\int e^x \sin x dx$

f) If $y=\displaystyle\int_{\tan x}^{0}\frac{dt}{1+t^2}$, find $\dfrac{dy}{dx}$.

7) Graph the integrand and use areas to evaluate the integral:

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

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8) Find the total area between the region and the *x*-axis:

$$y = x^3 - 3x^2 + 2x,\ 0 \le x \le 2$$

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9) Graph the given function $f(x) = 3x^2$ between $x = 0$ and $x = 1$. Then:

a) Using four rectangles whose height is given by the value of the function at the right-end of each subinterval, estimate the area under the graph.

b) Find the exact area under the graph.

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10) Suppose that *f* is a differentiable function shown in the accompanying graph and that the position at time *t* (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.

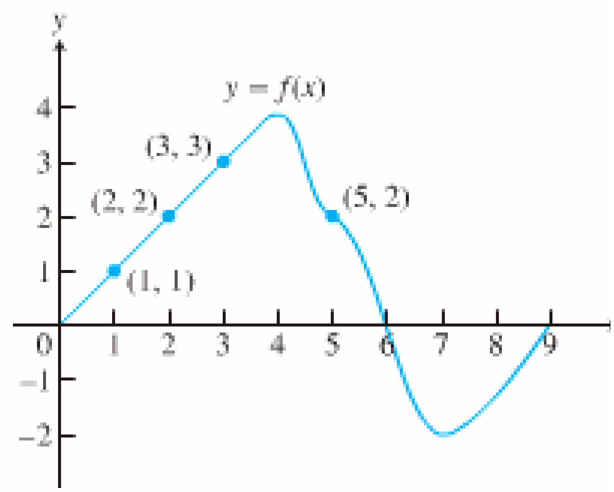
a) What is the particle's velocity at $t = 5$?

b) Is the acceleration at time $t = 5$ positive or negative?

c) What is the particle's position at time $t = 3$?

d) Approximately when is the acceleration zero?

e) On which side of the origin does the particle lie at time $t = 9$?



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11) A rectangle has its base on the *x*-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

TEST 2 – SOLUTIONS

(1a) if $m \le f(x) \le M$, $\forall x \in [a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

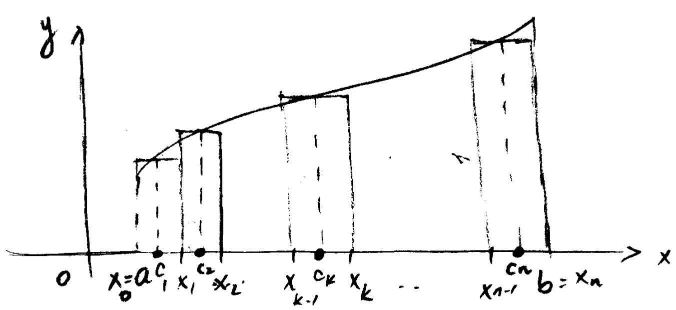
(b) $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$

where f: continuous fct. on $[a,b]$

$\Delta x = \frac{b-a}{n}$

$c_1, c_2, \ldots, c_n$ = sample points

$c_k \in [x_{k-1}, x_k]$



(c) c is a critical number for f if $c \in I$ and $f'(c) = 0$ or $f'(c)$ undefined.

(d) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

(e) An inflection point is a point where the graph of the function changes concavity.

(f) $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

(g) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

(h) $\int \frac{1}{x}\,dx = \ln|x| + C$ $(x \neq 0)$

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(2) $f(x) = \sqrt[3]{x^2 - x} = (x^2-x)^{1/3}$

Domain: $x \in \mathbb{R}$

$f'(x) = \frac{1}{3}(x^2-x)^{\frac{1}{3}-1}(2x-1)$

$f'(x) = \frac{2x-1}{3(x^2-x)^{2/3}}$

$f'(x) = 0$ iff $2x - 1 = 0$ iff $x = \frac{1}{2}$

$f'(x)$ undefined iff $x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

Critical numbers:

$x = 0,\ x = 1,\ x = \frac{1}{2}$

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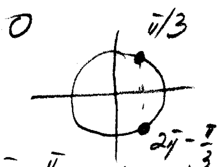
(3) $f(x) = x - 2\sin x$, $x \in [0, 3\pi]$

$f'(x) = 1 - 2\cos x$

$f'(x) = 0$ iff $1 - 2\cos x = 0$

$\cos x = \frac{1}{2}$



$x = \frac{\pi}{3} + 2\pi k$ or $x = 2\pi - \frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$

$k = 0$, $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

$k = 1$, $x = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

Critical numbers:

$x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$, $x = \frac{7\pi}{3}$

Evaluate f at the end-points and critical points:

$f(0) = 0 - 2\sin 0 = 0$

$f(3\pi) = 3\pi - 2\sin 3\pi = 3\pi$

$f(\frac{\pi}{3}) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - 2\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{3} - \sqrt{3}$

$f(\frac{5\pi}{3}) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{3} + \sqrt{3}$

$f(\frac{7\pi}{3}) = \frac{7\pi}{3} - 2\sin\frac{7\pi}{3} = \frac{7\pi}{3} - \sqrt{3}$

absolute min is $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$

absolute max is $f(3\pi) = 3\pi$

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(4) $f(x) = \dfrac{x}{(1+x)^2}$

x	$-\infty$		-1		0		1		2		∞
f'	$-$	$-$	$-$	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$
f	H.A. $y=0$	$\searrow$	$-\infty$ V.A	$-\infty$ $x=-1$	$\nearrow$ 0	$\nearrow$	$\frac{1}{4}$	$\searrow$	$\frac{2}{9}$	$\searrow$	H.A. $y=0$
f''	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$

(f) Domain: $x \in \mathbb{R} \setminus \{-1\}$

$\lim_{x\to\pm\infty} f(x) = \lim_{\pm\infty} \dfrac{\frac{x}{x^2}}{(\frac{1}{x}+1)^2} = \dfrac{0}{0+1} = 0$

so H.A $y = 0$

$\lim_{x\to -1} f(x) = \lim_{x\to-1} \dfrac{x}{(1+x)^2} = \dfrac{-1}{0^+} = -\infty$

so V.A. $x = -1$

x-I: $y = 0$ iff $x = 0$.

(f') $f'(x) = \dfrac{1\cdot(1+x)^2 - 2(1+x)(x)}{(1+x)^4}$

$f'(x) = \dfrac{(1+x)(1+x-2x)}{(1+x)^4} = \dfrac{(1+x)(1-x)}{(1+x)^4}$

$f'(x) = \dfrac{1-x}{(1+x)^3}$

$f'(x) = 0$ iff $1-x=0$, $x = 1$

$f(1) = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$

sign of $f'(x)$. TP $\begin{cases} x=-10, & y<0 \\ x = 0, & y>0 \\ x = 10, & y<0 \end{cases}$

(f'') $f''(x) = \dfrac{(-1)(1+x)^3 - 3(1+x)^2(1-x)}{(1+x)^6}$

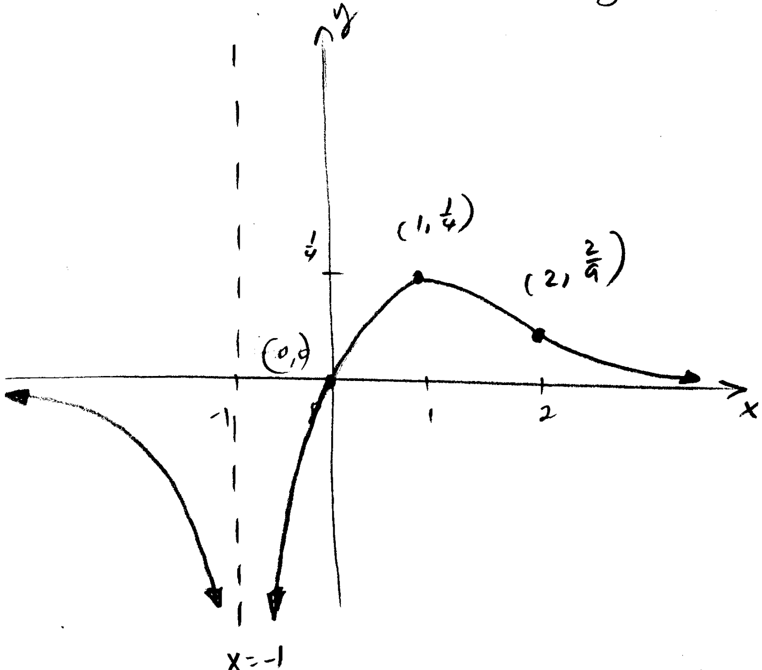
$f''(x) = \dfrac{(1+x)^2(-1-x-3(1-x))}{(1+x)^6}$

$f''(x) = \dfrac{-1-x-3+3x}{(1+x)^4} = \dfrac{2x-4}{(1+x)^4}$

$f''(x) = 0$ iff $2x-4=0$, $x=2$

$f(2) = \dfrac{2}{(1+2)^2} = \dfrac{2}{9}$

The sign of $f''(x)$ is given by the sign of $y = 2x-4$.



(b) absolute max / local max $= (1, \frac{1}{4})$
no minimum

(c) $(2, \frac{2}{9})$ = inflection point

(d) f is increasing on $(-1,1)$
f is decreasing on $(-\infty,-1)$
f is decreasing on $[1,\infty)$

(e) f is concave up on $[2,\infty)$
f is concave down on
$(-\infty,-1) \cup (-1,2)$

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(5) (a) $\lim_{x\to\infty}(\ln 2x - \ln(x+1)) =$

$= \lim_{\infty} \ln \frac{2x}{x+1}$

$= \ln \lim_{x\to\infty} \frac{2x}{x+1}$ $(\frac{\infty}{\infty}$ - l'Hopital$)$

$= \ln \lim_{x\to\infty} \frac{(2x)'}{(x+1)'}$

$= \ln \lim_{x\to\infty} \frac{2}{1}$

$= \ln 2$

(b) $\lim_{x\to 0}(1-2x)^{\frac{1}{x}} = 1^{\infty}$

$= \lim_{x\to 0} e^{\ln(1-2x)^{\frac{1}{x}}} =$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln(1-2x)}$

$= e^{\lim_{x\to 0}\frac{1}{x}\ln(1-2x)}$ (*)

$\lim_{x\to 0}\frac{1}{x}\ln(1-2x) =$

$= \lim_{x\to 0}\frac{\ln(1-2x)}{x} = \frac{0}{0}$ (l'Hopital)

$= \lim_{x\to 0}\frac{(\ln(1-2x))'}{x'}$

$= \lim_{x\to 0}\frac{\frac{1}{1-2x}(-2)}{1}$

$= \lim_{x\to 0}\frac{-2}{1-2x} = \frac{-2}{1} = -2$

Therefore (*)

$\lim_{x\to 0}(1-2x)^{\frac{1}{x}} = e^{-2}$

(c) $\lim_{x\to 0^+} \sqrt{x}\ln x = 0\cdot(-\infty)$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \frac{\infty}{\infty}$ (l'Hopital)

$= \lim_{0^+} \frac{(\ln x)'}{(x^{-\frac{1}{2}})'}$

$= \lim_{0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{1}{2}-1}}$

$= \lim_{0^+} \frac{-2x^{\frac{3}{2}}}{x} = \lim_{0^+} -2x^{\frac{3}{2}-1}$

$= (-2)\cdot 0 = 0$

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(6)

(a) $\int \ln x \, dx = x\ln x - \int \frac{1}{x}\cdot x \, dx$

integration by parts:

$\begin{bmatrix} f=\ln x & g'=1 \\ f'=\frac{1}{x} & g=x \end{bmatrix}$

$= x\ln x - \int dx$

$= x\ln x - x + C$

(b) $\int_{-1}^{3}(4+3x^3)\,dx = \int_{-1}^{3} 4\,dx + 3\int_{-1}^{3} x^3\,dx$

$= 4(3-(-1)) + 3\cdot\frac{x^4}{4}\Big]_{-1}^{3}$

$= 4(4) + \frac{3}{4}(3^4-(-1)^4)$

$= 16 + \frac{3}{4}(81-1) = 16 + \frac{3}{4}\cdot 80$

$= 16 + 3(20) = 76$

(c) $\int_1^2\left(\frac{1}{x} - e^{-x}\right)dx =$

$= \int_1^2 \frac{1}{x}\,dx - \int_1^2 e^{-x}\,dx$

$= \ln|x|\Big]_1^2 - \frac{e^{-x}}{-1}\Big]_1^2$

$= (\ln 2 - \ln 1) + e^{-x}\Big]_1^2$

$= \ln 2 - 0 + e^{-2} - e^{-1}$

$= \ln 2 + \frac{1}{e^2} - \frac{1}{e}$

(d) $\int\left(2^{t+1} - \sin t\right)dt =$

$= \int 2^{t+1}dt - \int \sin t\,dt$

$= \frac{2^{t+1}}{\ln 2} - (-\cos t) + C$

$= \frac{2^{t+1}}{\ln 2} + \cos t + C$

(e) let $I = \int e^x \sin x \, dx$

integration by parts.

$$\begin{cases} f = e^x & g' = \sin x \\ f' = e^x & g = -\cos x \end{cases}$$

$$I = -e^x \cos x - \int e^x(-\cos x)\, dx$$

$$I = -e^x \cos x + \int e^x \cos x \, dx$$

$$\begin{cases} f = e^x & g' = \cos x \\ f' = e^x & g = \sin x \end{cases}$$

$$I = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)$$

$$I = -e^x \cos x + e^x \sin x - I$$

$$2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{e^x(\sin x - \cos x)}{2} + C$$

(f) $y = \int_{\tan x}^{0} \frac{dt}{1+t^2}$

$$y = -\int_0^{\tan x} \frac{dt}{1+t^2}$$

$$\frac{dy}{dx} = -\frac{d}{dx} \int_0^{\tan x} \frac{dt}{1+t^2}$$

let $\tan x = u$

$$\frac{dy}{dx} = -\frac{d}{dx} \int_0^u \frac{dt}{1+t^2}$$

$$= -\left(\frac{d}{du} \int_0^u \frac{dt}{1+t^2}\right) \frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{1+\tan^2 x} \cdot \sec^2 x$$

$$\frac{dy}{dx} = \frac{-1}{\sec^2 x} \cdot \sec^2 x$$

$$\frac{dy}{dx} = -1$$

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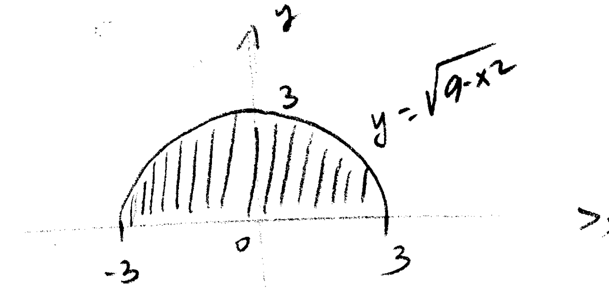
(7) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

let $y = \sqrt{9-x^2} \geq 0, \ \forall x \in [-3,3]$

upper semicircle

center $(0,0)$, $r=3$

$(y^2 = 9 - x^2, \quad x^2 + y^2 = 9)$



So, $\int_{-3}^{3} \sqrt{9-x^2}\, dx$ = area under the curve

$$= \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$$

(8) $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

X-in: $x^3 - 3x^2 + 2x = 0$

$x(x^2 - 3x + 2) = 0$

$x(x-2)(x-1) = 0$

$x = 0, \quad x = 1, \quad x = 2$

Area $= A = \left|\int_0^1 y\,dx\right| + \left|\int_1^2 y\,dx\right|$

$A = \left|\int_0^1 (x^3 - 3x^2 + 2x)\,dx\right| + \left|\int_1^2 (x^3 - 3x^2 + 2x)\,dx\right|$

$A = \left|\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1\right| + \left|\left[\frac{x^4}{4} - x^3 + x^2\right]_1^2\right|$

$A = \left|\left(\frac{1}{4} - 1 + 1\right) - 0\right| + \left|\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1}{4} - 1 + 1\right)\right|$

$A = \left|\frac{1}{4}\right| + \left|4 - 8 + 4 - \frac{1}{4}\right|$

$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

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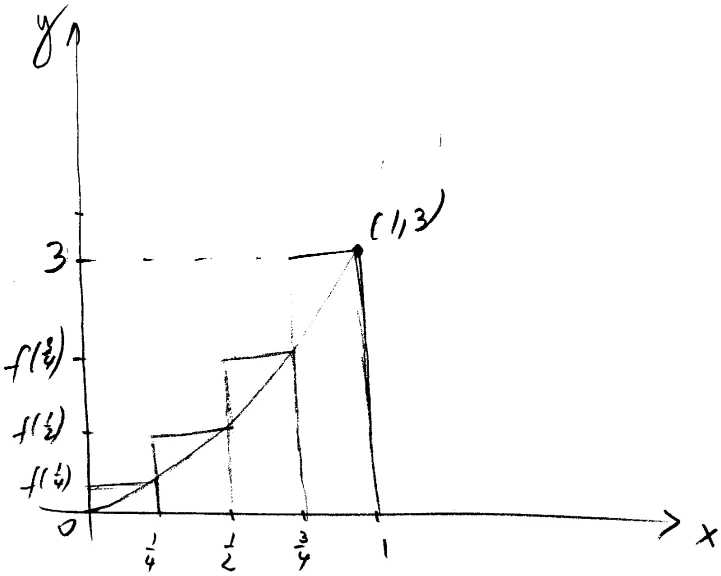
(9) $f(x) = 3x^2, \quad x \in [0,1)$



(a) Area = A

$A \approx \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1)$

$A \approx \frac{1}{4}\left(3\cdot\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{3}{4}\right)^2 + 3(1)^2\right)$

$= \frac{3}{4}\left(\frac{1}{16} + \frac{1}{4}^{\,4} + \frac{9}{16} + \frac{1}{1}^{\,16}\right)$

$= \frac{3}{4} \cdot \frac{1 + 4 + 9 + 16}{16} = \frac{3}{\cancel{4}_{2}} \cdot \frac{\cancel{30}^{15}}{16}$

$A \approx \frac{45}{32}$

OR

$A \approx \sum_{k=1}^{4} \frac{1}{4} f(x_k)$

$= \frac{1}{4} \sum_{k=1}^{4} 3x_k^2 = \frac{3}{4} \sum_{k=1}^{4} x_k^2$

$= \frac{3}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right)$

(b) $f(x) \ge 0, \quad \forall x \in [0,1]$

Therefore,

$A = \int_0^1 3x^2\,dx = 3\int_0^1 x^2\,dx$

$= 3 \cdot \frac{x^3}{3}\Big]_0^1 = x^3\Big]_0^1 = 1^3 - 0$

$A = 1$

(10) $s = \int_0^t f(x)\,dx$ position fct.

Then $\frac{ds}{dt}$ = velocity fct.

$$\frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)dx = f(t)$$

Therefore the graph $y = f(x)$ represents the velocity function

(a) $t=5$, $f(5) = 2$

(b) $\frac{d^2s}{dt^2} = \frac{df}{dt}$ = acceleration

At $t=5$ the slope of the tangent to the graph is negative, so the acceleration is negative

(c) $s = \int_0^3 f(x)\,dx$ = area under the graph between $x=0$ and $x=3$

$\int_0^3 f(x)\,dx = \frac{1}{2}\cdot 3\cdot 3 = 4.5$ m.

(d) acceleration is zero when the slope of the tangent to the curve is zero iff the tangent is horizontal.

At about $t=4$, $t=7$

(e) $s = \int_0^9 f(x)\,dx$ gives the position of the particle at $t=9$.

Because the area above the x-axis is greater than the area below the x-axis, $\int_0^9 f(x)\,dx > 0$ meaning that the particle is on the right side of the origin.

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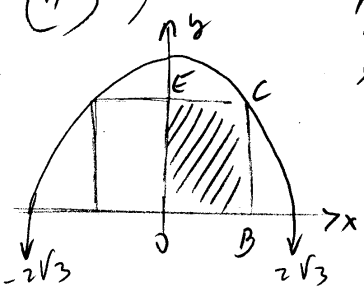
(11) $y = 12 - x^2$ parabola opening down with x-int: $(\pm 2\sqrt{3}, 0)$



The area of the rectangle inscribed is maximum iff Area (OBCE) = max

Let $A(x)$ = Area (OBCE)

$A(x) = x(12-x^2) = 12x - x^3$, $0 \le x \le 2\sqrt{3}$

$A'(x) = 12 - 3x^2 = 3(4-x^2)$

$A'(x) = 0$ iff $x = 2$ or $\cancel{x = -2}$ (not in $[0, 2\sqrt{3}]$)

Evaluate $A(x)$ at $x=0$, $x=2$, $x=2\sqrt{3}$

$A(0) = 0$

$A(2) = 2(12-4) = 32$ max.

$A(2\sqrt{3}) = 0$

So the maximum area occurs at $x=2$

The dimensions of the rectangle must be 4 units by $12 - 2^2 = 8$ units

max. area is 64 sq. units.