TEST #2 @ 150 points

Solve the problems on separate paper. Clearly label the problems. Show all steps in order to get credit. No proof, no credit given

1. Graph $f(x)=\sin x$ and $f^{-1}(x)=\sin^{-1}(x)$ on the same coordinate system, showing the relation between the two graphs (symmetry about the line $y=x$). Answer the following questions:

   a) What is the domain and range of $f(x)=\sin x$?

   b) What is the domain and range of $f^{-1}(x)=\sin^{-1}(x)$?

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2. a) Graph $y=1+3\sin(2x)$ between 0 and 2π. Identify the amplitude and period and label the axes accurately.

   b) Find the x-intercepts of the graph within the period graphed; that is, solve the equation $y=0$ in $[0,2\pi]$. Give exact answers as well as approximations.

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3. Graph $y=\frac{3}{4}\cos\left(2x+\frac{2\pi}{3}\right)$ over one period. Identify the amplitude, period, and phase shift and label the axes accurately.

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4. Find all real numbers x that satisfy each equation. Justify your answers.

   a) $\cos x=0$

   b) $\sin x=0$

   c) $\tan x=0$

   d) $\cot x=0$

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5. Evaluate the following. Give exact answers whenever possible.

   a) $\sin^{-1}\left(-\frac{1}{2}\right)$

   b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

   c) $\tan^{-1}\left(-\sqrt{3}\right)$

   d) $\cos^{-1}\left(\cos\frac{7\pi}{4}\right)$

   e) $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

   f) $\sin\left(\cos^{-1}\frac{1}{2}\right)$

6. Prove the following identities:

a) $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

b) $\dfrac{\cos a}{1-\tan a} + \dfrac{\sin a}{1-\cot a} = \sin a + \cos a$

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7. a) Find a formula for $\tan(a+b)$ in terms of tan a and tan b.

b) Find a formula for $\cos 3a$ in terms of $\cos a$.

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8. Solve the following equations.
When appropriate, show EXACT answers.
ONLY when NO exact answer is possible, express solutions rounded to two decimal places.

a) Find ALL solutions: $2\sin x - \sqrt{3} = 0$

b) Solve on $[0, 2\pi)$: $\cos(3x) = 1$

c) Solve on $[0, 2\pi)$: $2\tan\theta + 2 = 0$

d) Solve on $[0, 2\pi)$: $\sin\theta = -0.7$

e) Solve on $[0, 2\pi)$: $2\sin^2 x - \sin x - 1 = 0$

f) Find ALL solutions: $\sin 2\theta - \cos\theta = 0$

g) Solve on $[0, 2\pi)$: $2\sin^2 a - 2\cos a - 1 = 0$

(3) $y = \frac{3}{4}\cos\left(2x + \frac{2\pi}{3}\right)$

$y = \frac{3}{4}\cos 2\left(x + \frac{\pi}{3}\right)$

period $T = \frac{2\pi}{2} = \pi$

amplitude $A = \frac{3}{4}$

phase shift $= \frac{-\pi}{3}$

$[0,\pi] \longrightarrow \left[-\frac{\pi}{3}, \pi - \frac{\pi}{3}\right]$

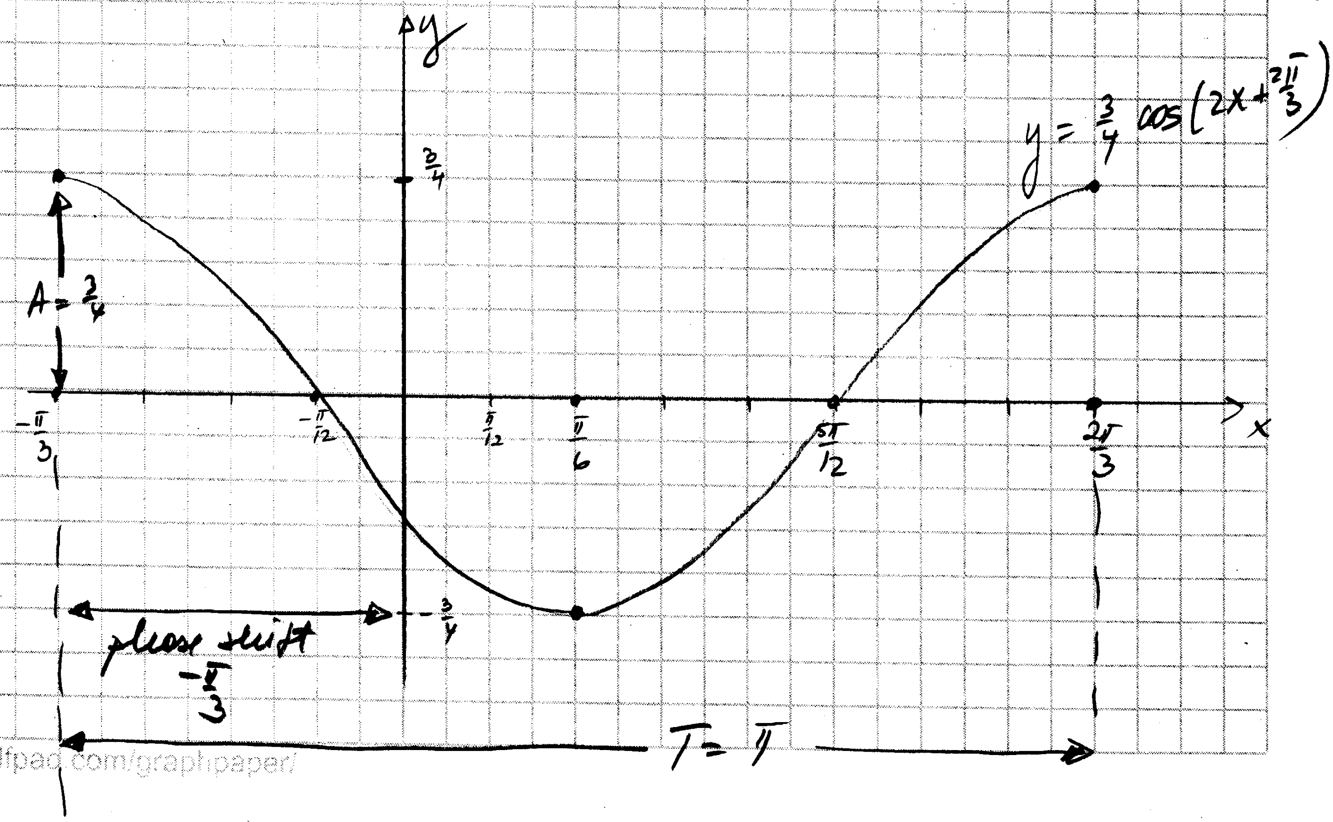
x-axis: $-\frac{\pi}{3}$

$-\frac{\pi}{3} + \frac{\pi}{4} = \frac{-\pi}{12}$

$\frac{-\pi}{12} + \frac{\pi}{4} = \frac{2\pi}{12} = \frac{\pi}{6}$

$\frac{2\pi}{12} + \frac{\pi}{4} = \frac{5\pi}{12}$

$\frac{5\pi}{12} + \frac{\pi}{4} = \frac{8\pi}{12} = \frac{2\pi}{3}$

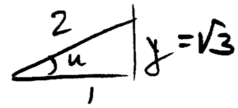


Method V

$$\sin\left(\cos^{-1}\tfrac{1}{2}\right) = ?$$

Let $\cos^{-1}\frac{1}{2} = u \in [0, \pi]$

then $\cos u = \frac{1}{2}$



$y = \sqrt{4-1} = \sqrt{3}$

then $\sin u = \sin\left(\cos^{-1}\frac{1}{2}\right) = \boxed{\frac{\sqrt{3}}{2}}$

⑥ ⓐ $$\cot\theta = \frac{\sin 2\theta}{1-\cos 2\theta}$$

Solution:

$$\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}$$

$$= \frac{\cancel{2\sin\theta}\cos\theta}{\cancel{2\sin^2\theta}}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

Therefor, the given equation is an identity

(b) $$\frac{\cos a}{1-\tan a} + \frac{\sin a}{1-\cot a} = \sin a + \cos a$$

Solution

$$\frac{\cos a}{1-\tan a} + \frac{\sin a}{1-\cot a} =$$

$$= \frac{\cos a}{1-\frac{\sin a}{\cos a}} + \frac{\sin a}{1-\frac{\cos a}{\sin a}}$$

$$= \frac{\cos^2 a}{\cos a-\sin a} + \frac{\sin^2 a}{\sin a-\cos a}$$

$$= \frac{\cos^2 a-\sin^2 a}{\cos a-\sin a}$$

$$= \frac{\cancel{(\cos a-\sin a)}(\cos a+\sin a)}{\cancel{\cos a-\sin a}}$$

$$= \cos a + \sin a$$

Therefor, the given equation is an identity.

(7) (a) $\tan(a+b) = \dfrac{\sin(a+b)}{\cos(a+b)}$

$$= \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$$

$$= \frac{\dfrac{\sin a \cancel{\cos b}}{\cos a \cancel{\cos b}} + \dfrac{\sin b \cancel{\cos a}}{\cancel{\cos a} \cos b}}{\dfrac{\cancel{\cos a} \cancel{\cos b}}{\cancel{\cos a} \cancel{\cos b}} - \dfrac{\sin a \sin b}{\cos a \cos b}}$$

$$= \frac{\dfrac{\sin a}{\cos a} + \dfrac{\sin b}{\cos b}}{1 - \dfrac{\sin a}{\cos a} \cdot \dfrac{\sin b}{\cos b}}$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

Therefore,

$$\boxed{\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}}$$

(b) $\cos 3a = \cos(a + 2a)$

$= \cos a \cos 2a - \sin a \sin 2a$

$= \cos a (2\cos^2 a - 1) - \sin a (2 \sin a \cos a)$

$= 2\cos^3 a - \cos a - 2\sin^2 a \cos a$

$= 2\cos^3 a - \cos a - 2\cos a (1 - \cos^2 a)$

$= 2\cos^3 a - \cos a - 2\cos a + 2\cos^3 a$

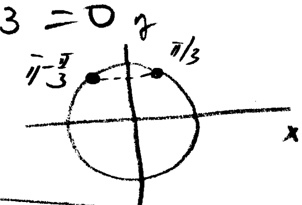
$= 4\cos^3 a - 3\cos a$

Therefore,

$$\boxed{\cos 3a = 4\cos^3 a - 3\cos a}$$

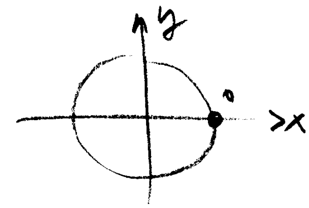
(8) (a) $2\sin x - \sqrt{3} = 0$,

$\sin x = \dfrac{\sqrt{3}}{2}$



$$\boxed{x = \frac{\pi}{3} + 2k\pi \text{ OR } x = \frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}}$$

(b) $\cos(3x) = 1$ in $[0, 2\pi)$



$3x = 2k\pi, \ k \in \mathbb{Z}$

$x = \frac{2}{3}k\pi, \ k \in \mathbb{Z}$

$k = 0, \quad x = 0 \in [0, 2\pi)$

$k = 1, \quad x = \frac{2\pi}{3} \in [0, 2\pi)$

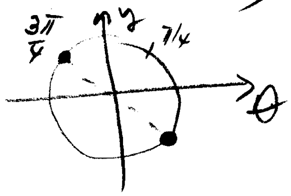
$k = 2, \quad x = \frac{4\pi}{3} \in [0, 2\pi)$

$k = 3, \quad x = 2\pi \notin [0, 2\pi)$

$$\boxed{x \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}}$$

(c) $2\tan\theta + 2 = 0$ in $[0, 2\pi)$

$\tan\theta = -1$



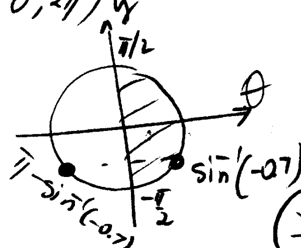
$\theta = \frac{3\pi}{4} + k\pi$, $k \in \mathbb{Z}$

$k = 0, \quad \theta = \frac{3\pi}{4}$

$k = 1, \quad \theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

$$\boxed{\theta \in \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}}$$

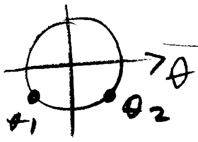
(d) $\sin\theta = -0.7$ in $[0, 2\pi)$



$$\begin{cases} \theta = \sin^{-1}(-0.7) + 2k\pi \\ \text{OR} \\ \theta = \pi - \sin^{-1}(-0.7) + 2k\pi \end{cases}$$

$k = 0, \quad \theta = \sin^{-1}(-0.7) \notin [0, 2\pi)$

or $\theta_1 = \pi - \sin^{-1}(-0.7) \approx 3.92$



$\in$ quadrant III

$k = 1, \quad \theta_2 = \sin^{-1}(-0.7) + 2\pi \approx 5.51$

$\in$ quadrant IV

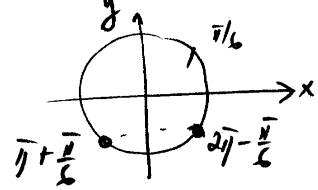
$$\boxed{\theta \in \{3.92, 5.51\}}$$

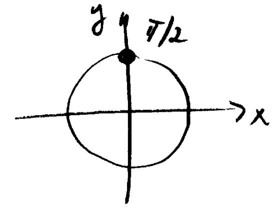
(e) $2\sin^2 x - \sin x - 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$2\sin x + 1 = 0$ OR $\sin x = 1$

$\sin x = -\frac{1}{2}$





$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$x = \frac{\pi}{2}$

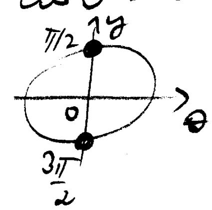
$$\boxed{x \in \left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$$

(f) $\sin 2\theta - \cos\theta = 0$ in $\mathbb{R}$

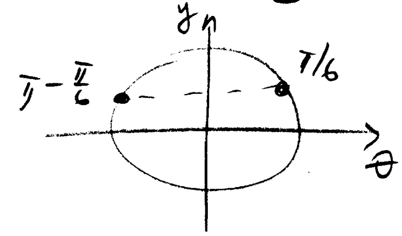
$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ OR $2\sin\theta - 1 = 0$



$\sin\theta = \frac{1}{2}$



$$\boxed{\theta = \frac{\pi}{2} + 2k\pi \text{ OR } \theta = \frac{3\pi}{2} + 2k\pi}$$

$$\boxed{\theta = \frac{\pi}{6} + 2k\pi \text{ OR } \theta = \frac{5\pi}{6} + 2k\pi}$$

$k \in \mathbb{Z}$

(9) $2\sin^2 a - 2\cos a - 1 = 0$ in $[0, 2\pi)$

$$2(1-\cos^2 a) - 2\cos a - 1 = 0$$

$$2 - 2\cos^2 a - 2\cos a - 1 = 0$$

$$-2\cos^2 a - 2\cos a + 1 = 0$$

$$2\cos^2 a + 2\cos a - 1 = 0$$

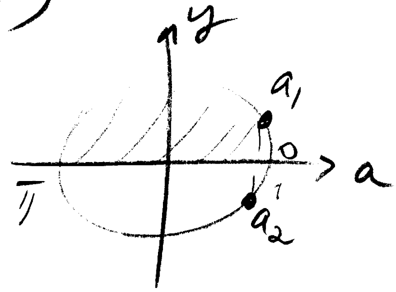
$$\cos a = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$\cos a = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4}$$

$$\cos a = \frac{-1 \pm \sqrt{3}}{2} \begin{cases} \approx 0.366 \\ \cancel{\approx -1.366} \end{cases}$$

not possible
$(\cos a \in [-1, 1])$

$\cos a = 0.366$



$a_1 = \cos^{-1}(0.366) \approx 1.20$

or

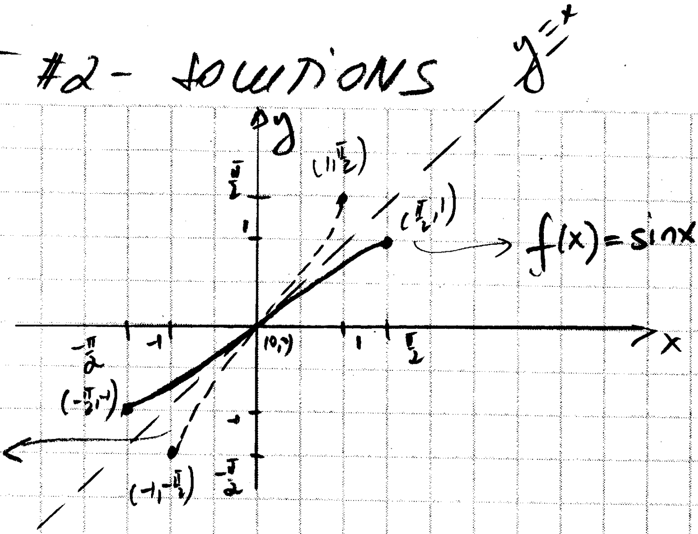
$a_2 = 2\pi - \cos^{-1}(0.366) \approx 5.09$

$$\boxed{a \in \{1.20,\ 5.09\}}$$

MATH150 TEST #2 - SOLUTIONS

(1) $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

$\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$



(2) (a) $y = 1 + 3\sin(2x)$

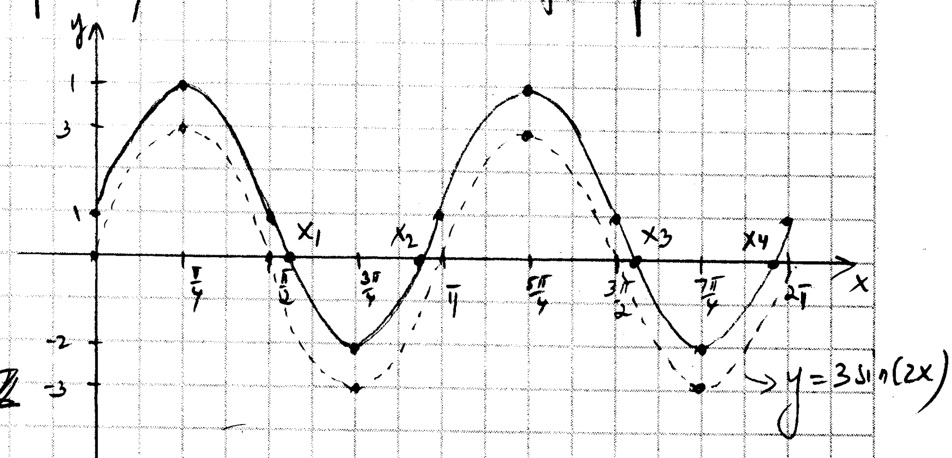
period $T = \frac{2\pi}{2} = \pi$
amplitude $A = 3$
vertical shifting up 1 unit

Take $[0, \pi]$, divide it into 4 equal parts (each of length $\frac{\pi}{4}$), sketch a sine curve of amplitude 3, then shift up 1 unit

(b) $1 + 3\sin(2x) = 0$

$3\sin(2x) = -1$

$\sin(2x) = \frac{-1}{3}$



$$\left\{\begin{array}{l} 2x = \sin^{-1}\left(-\frac{1}{3}\right) + 2k\pi \\ \quad \text{OR} \\ 2x = \pi - \sin^{-1}\left(-\frac{1}{3}\right) + 2k\pi \end{array}\right., k \in \mathbb{Z}$$

$$\left\{\begin{array}{l} x = \frac{1}{2}\sin^{-1}\left(\frac{-1}{3}\right) + k\pi \\ \text{OR} \\ x = \frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{3}\right) + k\pi \end{array}\right., k \in \mathbb{Z}$$

$$\left\{\begin{array}{l} x \approx -0.17 + k\pi \\ \quad \text{OR} \\ x \approx 1.74 + k\pi \end{array}\right., k \in \mathbb{Z}$$

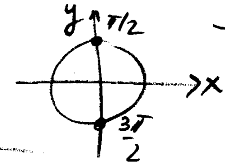
$k=0$: $\left\{\begin{array}{l} x = \frac{1}{2}\sin^{-1}\left(\frac{-1}{3}\right) \approx -0.17 \notin [0, 2\pi] \\ x_1 = \frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{3}\right) \approx 1.74 \in \text{quadrant II} \end{array}\right.$

$k=1$: $\left\{\begin{array}{l} x_2 = \frac{1}{2}\sin^{-1}\left(\frac{-1}{3}\right) + \pi \approx 2.97 \in \text{quad. II} \\ x_3 = \frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{3}\right) + \pi \approx 4.88 \in \text{quad. IV} \end{array}\right.$

$k=2$: $x_4 = \frac{1}{2}\sin^{-1}\left(\frac{-1}{3}\right) + 2\pi \approx 6.11 \in \text{quad. IV}$

x-∩ in $[0, 2\pi]$ are
$(1.74, 0), (2.97, 0), (4.88, 0), (6.11, 0)$

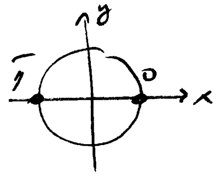
(4) (a) $\cos x = 0$



$$\left\{ \boxed{\begin{array}{l} x = \frac{\pi}{2} + 2k\pi \\ \text{OR} \\ x = \frac{3\pi}{2} + 2k\pi \end{array}, \; k \in \mathbb{Z}} \right.$$

(OR $x = \frac{\pi}{2} + k\pi$)

(b) $\sin x = 0$



$\boxed{x = k\pi, \; k \in \mathbb{Z}}$

(c) $\tan x = 0$ iff

$\frac{\sin x}{\cos x} = 0$ iff $\sin x = 0$

iff $\boxed{x = k\pi, \; k \in \mathbb{Z}}$

(d) $\cot x = 0$ iff

$\frac{\cos x}{\sin x} = 0$ iff $\cos x = 0$

iff $\boxed{\begin{array}{l} x = \frac{\pi}{2} + 2k\pi \\ \text{OR} \\ x = \frac{3\pi}{2} + 2k\pi \end{array}}$ $k \in \mathbb{Z}$

(5) Recall that

$\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1} : [-1,1] \to [0, \pi]$

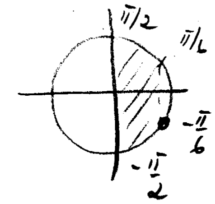
$\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{2}/2$
cos	$\sqrt{3}/2$	$\frac{1}{2}$	$\sqrt{2}/2$
tan	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1

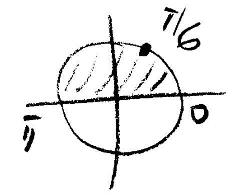
(a) $\sin^{-1}(-\frac{1}{2}) = \boxed{-\frac{\pi}{6}}$

b/c $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$

and $-\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



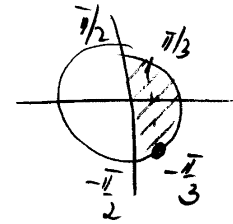
(b) $\cos^{-1}(\frac{\sqrt{3}}{2}) = \boxed{\frac{\pi}{6}}$



b/c $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

and $\frac{\pi}{6} \in [0, \pi]$

(c) $\tan^{-1}(-\sqrt{3}) = \boxed{-\frac{\pi}{3}}$



b/c $\tan(-\frac{\pi}{3}) = -\sqrt{3}$

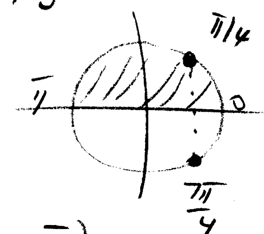
and $-\frac{\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(d) $\cos^{-1}(\cos\frac{7\pi}{4}) \neq \frac{7\pi}{4}$

b/c $\frac{7\pi}{4} \notin [0, \pi]$

But,

$\cos(\frac{7\pi}{4}) = \cos\frac{\pi}{4}$



Therefore,

$\cos^{-1}(\cos\frac{7\pi}{4}) = \cos^{-1}(\cos\frac{\pi}{4})$

$= \boxed{\frac{\pi}{4}}$

(e) $\sin(\sin^{-1}\frac{\sqrt{2}}{2}) = \boxed{\frac{\sqrt{2}}{2}}$

(f) $\sin(\cos^{-1}\frac{1}{2})$

Method I $\sin(\cos^{-1}\frac{1}{2}) =$

$= \sin(\frac{\pi}{3})$

$= \boxed{\frac{\sqrt{3}}{2}}$